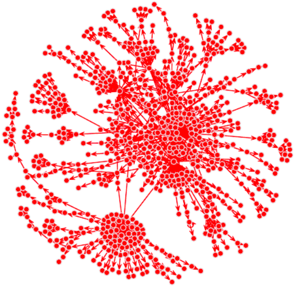
Networks and Games - Complex systems with structural interactions and epidemic processes

Yezekael Hayel
joint works with S. Trajanovski, F. Kuipers, E. Altman, P. Van Mieghem

17th April 2019, NCPG Workshop, IHP

Outline

Motivation



Controlling epidemic processes on networks is very important for many applications:

1. computer networks security,
2. information/rumor spreading,
3. failures propagation,
4. etc

Motivation

Many techniques are studied to manage epidemics:

1. immunization,
2. quarantine,
3. anti-spyware
4. etc

Our first problem is to study individual protection strategies (investment) against epidemics on a network architecture, from several perspectives:

1. users are selfish (protect themself),
2. positive externalities (by protecting himself, a node protects his neighbor),
3. full description of the (equilibrium) protection strategy depending on the network architecture (graph topology).

Motivation

A second problem is the design of an optimal network topology that would be:

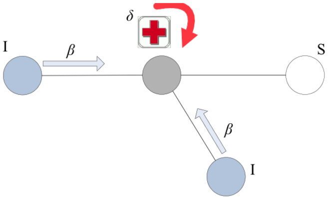
1. secure and resilient to viruses,
2. high performance.

The two objectives are conflicting.
Objectives and challenges:

1. design in a decentralized way (selfish agents)
2. what is the potential lost (e.g., PoA)?
3. different performance measures

Outline

Virus-spread model



The probability of infection of any node depends on the (stochastic) state of its neighbors. The Susceptible-Infected-Susceptible (SIS) model:

$$\frac{dv_i(t)}{dt} = \beta(1 - v_i(t)) \sum_{j=1}^{N} a_{ij} v_j(t) - \delta v_i(t) \tag{1}$$

where

- $v_i(t)$ is the infection probability of node i at time t
- $a_{ij} = 1$ nodes i and j are directly connected and $a_{ij} = 0$, otherwise
- a node can infect its direct healthy neighbors with rate β
- a infected node can be cured at rate δ

Virus-spread model

Main interest

The infection probability $v_{i\infty}$ of node i in the *metastable regime*

$$0 = \beta \left(1 - v_{i\infty}\right) \sum_{j=1}^{N} a_{ij} v_{j\infty} - \delta v_{i\infty}$$

or re-written

$$v_{i\infty} = 1 - \frac{1}{1 + \tau \sum_{j=1}^{N} a_{ij} v_{j\infty}}$$

where

- $\tau = \frac{\beta}{\delta}$ is called the *effective infection* rate.
- The epidemic threshold τ_c is defined as a value of τ, such that $v_{i\infty} > 0$ if $\tau > \tau_c$, and otherwise $v_{i\infty} = 0$ for all $i \in \{1, 2, \ldots, N\}$

Outline

Decentralized Protection

The infection probabilities could be substantially different after some nodes decide to invest in a protection, causing those nodes not to be part of the epidemic process (node and associated links disappear).

We consider different network interaction topologies:

- single community network: which could be regarded as a simple social network or a wireless and other full mesh networks (e.g., MANETs),
- multi-communities network: composed of M cliques/communities all interconnected to any other community by a *core node*.

Decentralized Protection

Complete graph: We consider a complete graph K_n with n nodes. By symmetry, we have for each node i in a complete graph:

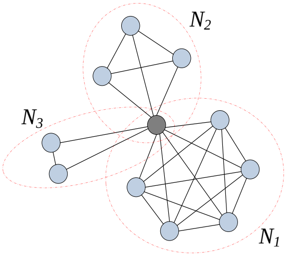
$$v_{\cdot,\infty}(n) = v_{i,\infty}(n) = \begin{cases} 1 - \frac{1}{\tau(n-1)}, & \text{if} \quad \tau \geq \frac{1}{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Decentralized Protection

Multi-community graph: The core node functions as a bridge between all communities: the M fully connected graphs are interconnected through one common node. We obtain for $\forall m = 1, 2, \ldots, M$,

$$v_{\infty}^{(\mathcal{N}_m)}(n_m, u_{\infty}) = 1 - \frac{1}{1 + \tau_m (n_m - 1) v_{\infty}^{(\mathcal{N}_m)} + \tau_m u_{\infty}} \qquad (2)$$

where $v_{\infty}^{(\mathcal{N}_m)}$ is the metastable state infection probability for any non-core node of community $\mathcal{N}_m$ and u_{∞} is the infection probability for the core node.



Individual protection strategy

In the investment game on the complete graph K_N, each node is a player and decides individually to invest in antivirus protection (protect himself).

- the investment cost is C,
- the infection cost is H,
- If a player decides not to invest, his cost is a linear function of its infection probability $v_{i,\infty}(n)$ of node i in the metastable state of the SIS process,i.e.

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E[X_i(n;t)]dt = v_{i,\infty}(n).$$

- The decisions of all the nodes induce an overlay graph only composed of the nodes that have decided *not* to invest.

Individual protection strategy

We have several properties for our non-cooperative game.

- The payoff S_{i1} of a player $i \in \{1, 2, \ldots, N\}$ which decides to invest is defined by: $S_{i1} = C := S_1$.
- The payoff of a player i which decides not to invest is: $S_{i0}(n) = Hv_{i,\infty}(n) := S_0(n)$.

Individual protection strategy

We have several properties for our non-cooperative game.

- The payoff S_{i1} of a player $i \in \{1, 2, \dots, N\}$ which decides to invest is defined by: $S_{i1} = C := S_1$.
- The payoff of a player i which decides not to invest is: $S_{i0}(n) = Hv_{i,\infty}(n) := S_0(n)$.
- The payoff of a player depends on the number of players that choose his action, then it is a congestion (potential) game.
- The game is *potential*, where $\Phi(n) = C(N-n) + H\sum_{k=2}^{n} v_{\cdot,\infty}(k)$ is the potential function of the game.

Individual protection strategy: Complete graph

At a Nash equilibrium, no node has an interest to change unilaterally his decision. The number n^* of nodes that do not invest at a Nash equilibrium is defined for any player i, by: $S_{i1} \leq S_{i0}(n^* + 1)$ and $S_{i0}(n^*) \leq S_{i1}$.

Individual protection strategy: Complete graph

At a Nash equilibrium, no node has an interest to change unilaterally his decision. The number n^* of nodes that do not invest at a Nash equilibrium is defined for any player i, by: $S_{i1} \leq S_{i0}(n^*+1)$ and $S_{i0}(n^*) \leq S_{i1}$.
For the number of nodes n^* that do not invest at equilibrium, the following inequality holds:

$$v_\infty(n^*) \leq \frac{C}{H} \leq v_\infty(n^*+1).$$

Moreover, above the epidemic threshold ($\tau > \frac{1}{N-1}$), n^* is uniquely defined by:

$$n^* = \begin{cases} \min\left\{N, \lceil \frac{1}{(1-\frac{C}{H})\tau} \rceil\right\}, & \text{if } C < H \\ N, & \text{otherwise} \end{cases}$$

where $\lceil x \rceil$ is the closest integer greater or equal than x and N is the total number of nodes.

Individual protection strategy: Complete graph

We propose a simple fully decentralized Reinforcement Learning Algorithm (RLA) that converges to a pure Nash Equilibrium in our invest game.

- At each discrete time slot k, independently, each node i decides whether to invest in antivirus protection.
- We denote by $\bar{\sigma}[k] = (\sigma_1[k], \ldots, \sigma_N[k])$ the vector of pure actions of all the nodes at time k.
- The pure action $\sigma_i[k]$ of node i at time slot k is an element from $\{0, 1\}$, where action 1 means node i invests and action 0 otherwise.
- The probability that node i invests at time slot k (i.e. $\sigma_i[k] = 1$) is denoted by $p_i[k] = \Pr[\sigma_i[k] = 1]$.

Individual protection strategy: Complete graph

1. Set an initial probability $p_i[0]$, for each user $i \in \{1, \dots, N\}$.
2. At every time slot k, each node i invests with probability $p_i[k] = \Pr[\sigma_i[k] = 1]$, which determines its pure action $\sigma_i[k]$.
3. Each player i has a negative utility (cost) $S_{i\sigma_i[k]}(n[k])$, which is equal to:

$$S_{i\sigma_i[k]}(n[k]) = \begin{cases} C, & \text{if} \quad \sigma_i[k] = 1, \\ Hv_{i,\infty}(n[k]), & \text{otherwise.} \end{cases}$$

   where $n[k] = N - \sum_{j=1}^{N} \sigma_j[k]$.

4. The cost of each node i is normalized: $\tilde{S}_{i\sigma_i[k]}(n[k]) = \frac{S_{i,\sigma_i[k]}(n[k])}{C+H}$.
5. Each node i updates its probability according to the following rule:

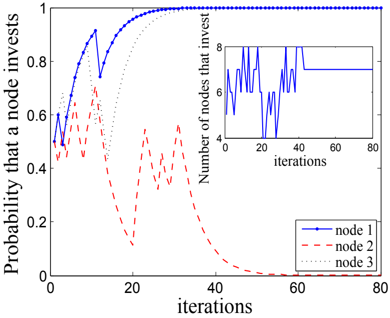
$$p_i[k+1] \leftarrow p_i[k] + b[k]\tilde{S}_{i\sigma_i[k]}(n[k])(\sigma_i[k] - p_i[k]),$$

   where $b[k]$ is the learning rate.

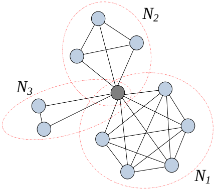
6. Stop when a stopping criterion is met (for example, the maximum of the differences between consecutive updates is smaller than a small ε); else increase k by 1 and go to step 2).

Individual protection strategy: Complete graph

- Any potential game possesses a Lyapunov function F.
- The decentralized algorithm converges almost surely to a pure (ε-)Nash equilibrium.
- The convergence time $T \leq O(\frac{F(\sigma[0])}{\varepsilon})$ only depends on ε and the Lyapunov value $F(\sigma[0])$ of the initial strategy $\sigma[0]$.
- The algorithm is simple and fully distributed, the only required information for each node is its instantaneous cost at each time slot.

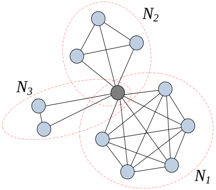


Individual protection strategy: Multi-community graph



Assuming that the infection probability u_∞ of the core node is given, our game with M communities is equivalent to M independent potential games.

Individual protection strategy: Multi-community graph

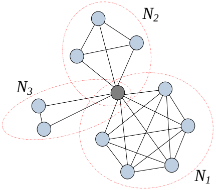


Assuming that the infection probability u_∞ of the core node is given, our game with M communities is equivalent to M independent potential games. The infection probability $v_\infty^{(\mathcal{N}_m)}$ of a non-core node in community $\mathcal{N}_m$ depends only on n_m (size of community N_m),

$$v_\infty^{(\mathcal{N}_m)}(n_m, u_\infty) = \frac{V(\tau_m, n_m, u_\infty)\left(1 + \sqrt{1 + \frac{4\tau_m^2 u_\infty (n_m - 1)}{V(\tau_m, n_m, u_\infty)}}\right)}{2\tau_m(n_m - 1)} \tag{3}$$

where $V(\tau_m, n_m, u_\infty) = \tau_m(n_m - 1) - \tau_m u_\infty - 1$.

Individual protection strategy: Multi-community graph



Assuming that the infection probability u_∞ of the core node is given, our game with M communities is equivalent to M independent potential games. The infection probability $v_\infty^{(\mathcal{N}_m)}$ of a non-core node in community $\mathcal{N}_m$ depends only on n_m (size of community N_m),

$$v_\infty^{(\mathcal{N}_m)}(n_m, u_\infty) = \frac{V(\tau_m, n_m, u_\infty)\left(1 + \sqrt{1 + \frac{4\tau_m^2 u_\infty (n_m - 1)}{V(\tau_m, n_m, u_\infty)}}\right)}{2\tau_m(n_m - 1)} \qquad (3)$$

where $V(\tau_m, n_m, u_\infty) = \tau_m(n_m - 1) - \tau_m u_\infty - 1$. Then the equation:

$$u_\infty = 1 - \frac{1}{1 + \sum_{m=1}^{M} \tau_m n_m v_\infty^{(\mathcal{N}_m)}},$$

has a unique solution in $(0, 1)$.

Individual protection strategy: Multi-community graph

Iterative heuristic procedure to compute a pure Nash equilibrium of this parametric potential game:

1. Fixed an initial value for $u_\infty[0]$.

Individual protection strategy: Multi-community graph

Iterative heuristic procedure to compute a pure Nash equilibrium of this parametric potential game:

1. Fixed an initial value for $u_\infty[0]$.
2. Based on this value, we solve the M independent potential games and we obtain the solution vector $\mathbf{n}^*(u_\infty[k]) = (n_1^*(u_\infty[k]), \ldots, n_M^*(u_\infty[k]))$. We denote for each community $\mathcal{N}_m$, the following parametric potential function: $\Phi_m(n_m, N_m, u_\infty[k]) = C(N_m - n_m) + H \sum_{i=2}^{n_m} v_\infty^{(\mathcal{N}_m)}(i, u_\infty[k])$. Hence, $n_m^*(u_\infty[k]) = \arg\min_{n_m} \Phi_m(n_m, N_m, u_\infty[k])$ for all m.

Individual protection strategy: Multi-community graph

Iterative heuristic procedure to compute a pure Nash equilibrium of this parametric potential game:

1. Fixed an initial value for $u_\infty[0]$.
2. Based on this value, we solve the M independent potential games and we obtain the solution vector $\mathbf{n}^*(u_\infty[k]) = (n_1^*(u_\infty[k]), \ldots, n_M^*(u_\infty[k]))$. We denote for each community $\mathcal{N}_m$, the following parametric potential function: $\Phi_m(n_m, N_m, u_\infty[k]) = C(N_m - n_m) + H\sum_{i=2}^{n_m} v_\infty^{(\mathcal{N}_m)}(i, u_\infty[k])$. Hence, $n_m^*(u_\infty[k]) = \arg\min_{n_m} \Phi_m(n_m, N_m, u_\infty[k])$ for all m.
3. Further, we compute the infection probability of a node from community m by the function $v_\infty^{(\mathcal{N}_m)}(n_m^*(u_\infty[k]), u_\infty[k])[k]$ from equation (3).
4. We update the infection probability of the core node: $u_\infty[k+1] = 1 - \frac{1}{1+\sum_{m=1}^{M} \tau_m n_m^*(u_\infty[k]) v_\infty^{(\mathcal{N}_m)}(n_m^*(u_\infty[k]), u_\infty[k])[k]}$.

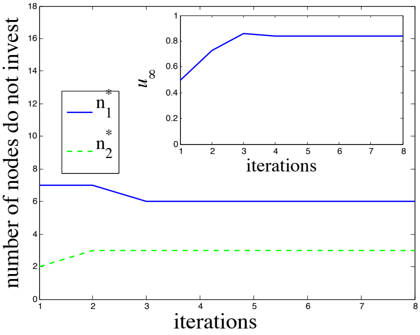
Individual protection strategy: Multi-community graph

Iterative heuristic procedure to compute a pure Nash equilibrium of this parametric potential game:

1. Fixed an initial value for $u_\infty[0]$.
2. Based on this value, we solve the M independent potential games and we obtain the solution vector $\mathbf{n}^*(u_\infty[k]) = (n_1^*(u_\infty[k]), \ldots, n_M^*(u_\infty[k]))$. We denote for each community $\mathcal{N}_m$, the following parametric potential function: $\Phi_m(n_m, N_m, u_\infty[k]) = C(N_m - n_m) + H \sum_{i=2}^{n_m} v_\infty^{(\mathcal{N}_m)}(i, u_\infty[k])$. Hence, $n_m^*(u_\infty[k]) = \arg\min_{n_m} \Phi_m(n_m, N_m, u_\infty[k])$ for all m.
3. Further, we compute the infection probability of a node from community m by the function $v_\infty^{(\mathcal{N}_m)}(n_m^*(u_\infty[k]), u_\infty[k])[k]$ from equation (3).
4. We update the infection probability of the core node: $u_\infty[k+1] = 1 - \frac{1}{1 + \sum_{m=1}^{M} \tau_m n_m^*(u_\infty[k]) v_\infty^{(\mathcal{N}_m)}(n_m^*(u_\infty[k]), u_\infty[k])[k]}$.
5. Stop if $\mid u_\infty[k+1] - u_\infty[k] \mid < \varepsilon$ and thus $u_\infty = u_\infty[k+1]$, otherwise increase $k \leftarrow k+1$ and start with step 2).

Individual protection strategy: Multi-community graph

We consider an example with $M = 2$ communities with $N_1 = 10$, $N_2 = 15$, $\tau_1 = 0.5$ and $\tau_2 = 1.5$. Second, we consider the following stopping criteria $\varepsilon = 10^{-7}$ and we observe that the number of iterations to achieve an equilibrium is very small (8 iterations).



Outline

Game-formation model

The game model is defined by the following:

- Player i (a node) aims to minimize its own *cost function* J_i
- The global *social cost* J is defined as $J = \sum_{i=1}^{N} J_i$
- **Goal**: existence and characterization of (pure) *Nash Equilibria*
- Efficiency versus the global optimum

$$\text{PoA} = \frac{J(\text{worst NE})}{\min J}, \ \text{PoS} = \frac{J(\text{best NE})}{\min J}$$

Game-formation model

Virus Spread-Cost **(VSC)** network formation game.
Each node determines the links with other nodes. The utility J_i of player i is a weighted sum:

- the infection probability $v_{i\infty}$,
- and the cost αk_i of all the links that i installs.

Condition: Player i should be able to reach all the nodes in the network.

The utility of player i is given by:

$$J_i = \begin{cases} \alpha \cdot k_i + v_{i\infty}, & \text{if } i \text{ can reach all the nodes,} \\ \infty, & \text{otherwise.} \end{cases}$$

Results

***Virus Spread-Cost* (VSC).** The social cost J for the whole network is:

$$J = \alpha \sum_{i=1}^{N} k_i + \sum_{i=1}^{N} v_{i\infty} = \alpha L + \sum_{i=1}^{N} v_{i\infty}$$

if the graph is connected, otherwise $J = \infty$, where

- $L = \sum_{i=1}^{N} k_i$ is the number of links in the network

Lemma

The infection probability $v_{i\infty}(G)$ of each node i in the metastable state in network G is not bigger than the infection probability $v_{i\infty}(G+l)$ of node i in the metastable state in network $G+l$ obtained by adding a link l to G.

Proof.

Based on the canonical form of $v_{i\infty}$. □

Results

***Virus Spread-Cost* (VSC)**. We look for the possible Nash Equilibria.

Theorem

If a Nash Equilibrium is achieved, then the constructed graph is a tree.

Proof.

By contradiction. □

Observation

A Nash Equilibrium is achieved for both the star graph and the path graph, but not all trees are Nash Equilibria.

Proof.

"In both directions" for the *star* graph and the *path*, and by counter examples for other *trees*. □

Results

Theorem

For low values of the effective infection rate $\tau \leq \tau_c(K_{1,N-1})$, the social cost $J(T) = \alpha(N-1)$ for any tree T. For values of the epidemic threshold $\tau_c(K_{1,N-1}) < \tau \leq \tau_c(P_N)$, the social cost $J(T) > J(P_N) = \alpha(N-1)$ for any tree T.

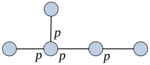
Proof.

Using a spectral approach and the result of [Lovász and Pelikán, 1973]. □

Observation

There are values of τ such that worst- and best-case Nash Equilibria are achieved for trees different from star $K_{1,N-1}$ and path P_N.

Proof.



The example is in the figure. For $\tau \in [1.475, 1.589]$, the tree is the best-case Nash Equilibrium and has optimal social cost. □

Results

***Virus Spread-Cost* (VSC)**.

Corollary

For both high and low effective infection rate , $PoS = 1$ and $PoA = \max\{\frac{J(P_N)}{J(K_{1,N-1})}, \frac{J(K_{1,N-1})}{J(P_N)}\}$.

Corollary

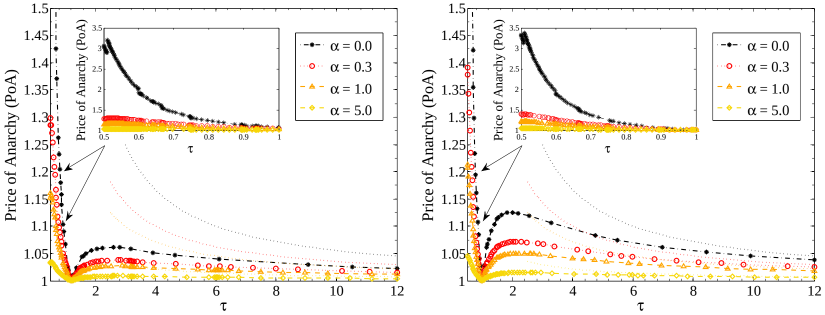
For sufficiently high effective infection rate τ, in the virus spread-cost game formation,

$$PoA < 1 + \frac{1}{2\big(\tau(\alpha+1) - 1\big)},$$

where $\tau(\alpha+1) > 1$.

Results

***Virus Spread-Cost* (VSC).**



(a) $N = 10$.   (b) $N = 1000$.

Figure: The Price of Anarchy (PoA). The value of N does not influence much, although higher N, in Figure 1b, implies more noticeable difference for various α. Dotted lines represent the bound from Corollary 5.

Results

Pairwise Nash Equilibrium (A PNE exists if and only if there is an interest of both nodes for having a link between them) and pairwise stability (The pairwise stability determines if each player is robust to one-link deviations, not necessarily installed by him, from a unilateral move).

Theorem

In the VSC *game, a Nash Equilibrium implies pairwise stability and a Nash Equilibrium is equivalent to a pairwise Nash Equilibrium.*

Outline

Conclusions

Decentralized Protection Strategies **(DPS)**.

- Interaction structure defines the playground of a game.
- The network is also the support of an epidemic process.
- Complete description of the pure Nash equilibrium.
- Extension of the potential game concept for multicommunity graphs.
- Description of RLA that converges to the pure Nash equilibrium.

Conclusions

***Virus Spread-Cost* (VSC)**.

- We find that a Nash Equilibrium always exists and is a *tree*, although not any tree is a Nash Equilibrium. The tree that is the worst-case Nash Equilibrium depends on the effective infection rate.
- A Nash Equilibrium in this game is shown to be *pairwise stable* [Myerson91]
- For high effective infection rate τ, the Price of Anarchy (PoA) in the VSC game is generally close to 1, independent from the number of hosts, and is inversely proportional to the virus infection rate and the link installation cost. This implies that non-cooperative players still form a close-to-optimal topology for high τ.
- On the other hand, PoA may be very high for small effective infection rate τ and/or small installation cost.
- Future work: mixed Equilibria, player coalitions, inhomogeneous costs weights, time-varying networks ...

Questions?