Atomic Congestion Games with Stochastic Demand — Convergence and Price-of-Anarchy

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Network, Population and Congestion Games
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You are planning your commute route for tomorrow.

Not sure about your departure time, nor who might be on the road.

A game with a random set of players !

Outline – Convergence

Congestion games model strategic situations that feature crowding externalities, where costs are monotone in the number of players.

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Can we precise in which sense the discrete and continuous models are close ?

This depends on what we mean by *"small"*... For instance:

- Player i has a small load $w_i \sim 0$ to be transported with certainty,
- Player i has a unit load but is present with small probability $p_i \sim 0$.

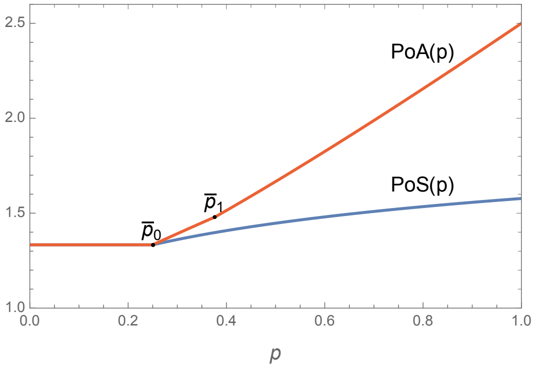
Depending on which limit we consider, we get different nonatomic models.

Outline – Price of Anarchy

Focusing on atomic congestion games with *affine costs and stochastic demands* where each player is present with probability p, we study how

$$\mathrm{PoA}(p) = \frac{\text{Social cost of worst equilibrium}}{\text{Least possible social cost}}$$

as a function of p.



1. Introduction

2. Convergence of Congestion Games
   - Nonatomic games and Wardrop equilibria
   - Weighted atomic games: convergence of Nash equilibria
   - Stochastic atomic games: convergence of Bayes-Nash equilibria

3. Price-of-Anarchy for Stochastic ACGs with Affine Costs
   - Upper bounds
   - Lower bounds
   - Price-of-Stability

Congestion Games

A *congestion game* is described by

- a set of *resources* $e \in E$ with continuous monotone costs $c_e : \mathbb{R} \to \mathbb{R}_+$
- a set of *types* $t \in T$ with corresponding strategy sets $S_t \subseteq 2^E$
- a (continuous or discrete) *demand* $d_t \geq 0$ for each type $t \in T$

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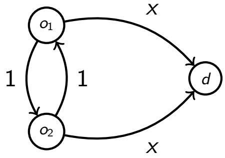
Example: In routing games the resources are the edges of a network $G = (V, E)$, T is the set of OD pairs, and S_t the paths connecting (o_t, d_t).



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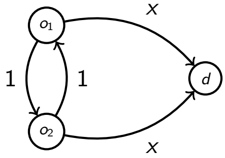

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ATOMIC SPLITTABLE: continuous, few players $\to$ fluids, sand, telecom

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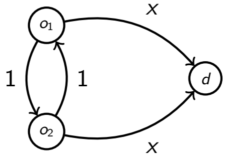

NON-ATOMIC: continuous, fine grained, many players → urban traffic
ATOMIC SPLITTABLE: continuous, few players → fluids, sand, telecom

ATOMIC UNSPLITTABLE: discrete, few players → vessels, airplanes
STOCHASTIC: unpredictable → packets or vehicles over a network

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where $x_e = \sum_{s \ni e} y_s$ *are the induced resource-loads.*

Atomic Splittable Congestion Games

Atomic splittable congestion games are similar to non-atomic models in that demands are continuous and can be split arbitrarily over different strategies.

The fundamental differences are:

1. There are finitely many players, each one controls a fraction of the demand.
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For the precise statement and subsequent generalizations, see Jacquot & Wang (2018) and references therein.

Here we address the discrete cases: unsplittable and stochastic demands.

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Given mixed strategies $\pi_i \in \Delta(S_{t_i})$, the Bernoulli random variables $X_{i,e}$ are independent across players with $\mathbb{P}(X_{i,e} = 1) = \sum_{s_i \ni e} \pi_i(s_i)$.

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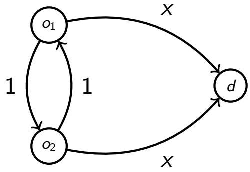
Example. A 2-player routing game with equal weights $w_i = 1$



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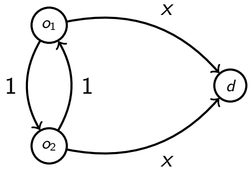

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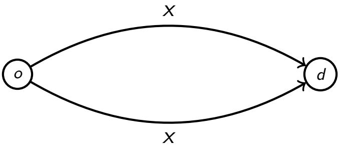
Example: Routing 10 players over 2 identical parallel links.



Wardrop Convergence for Vanishing Weights

Theorem

Consider an arbitrary sequence π^n of mixed equilibria for a sequence of weighted ACGs with player sets $N = \{1, \ldots, n\}$ and weights w_i^n such that

$$\begin{cases} a) & \max_{i \in N} w_i^n \to 0 \\ b) & (\forall t \in T) \;\; d_t^n \triangleq \sum_{i : t_i^n = t} w_i^n \to d_t \end{cases}$$

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1. *The sequence y^n of expected strategy loads $y_s^n = \sum_i w_i^n \pi_i^n(s)$ is bounded and each accumulation point $\bar{y}$ is a Wardrop equilibrium for the nonatomic game with demands d_t and costs $c_e(\cdot)$.*

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Remark: If the $c_e(\cdot)$'s are strictly monotone, then $\bar{x}$ is unique and $W_e^n \xrightarrow{\mathscr{D}} \bar{x}_e$.

Stochastic Atomic Congestion Games

A *stochastic atomic congestion game* features finitely many players $i \in N$ with types $t_i \in T$, unit weights $w_i = 1$, and a probability of being active

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REMARK. The costs $c_e(\cdot)$ need only be defined over the integers $c_e : \mathbb{N} \to \mathbb{R}_+$, and the continuity assumption becomes irrelevant.

Stochastic ACGs are Potential Games

Theorem

Every stochastic ACG is a potential game, hence it has pure Nash equilibria, with potential given by

$$\Phi(s) \triangleq \mathbb{E}\left[\sum_{e\in E}\sum_{k=1}^{N_e(s)} c_e(k)\right]$$

where $N_e(s) = \sum_{i\in N} Y_i \mathbb{1}_{\{e\in s_i\}}$.

Poisson Convergence for Vanishing Probabilities

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Let π^n be an arbitrary sequence of Bayes-Nash equilibria for a sequence of stochastic ACGs with unit weights $w_i = 1$ and probabilities p_i^n such that

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Suppose further that $\mathbb{E}[X^2 c_e(1+X)] < \infty$ for every $X \sim \mathrm{Poisson}(x)$, and set

$$\tilde{c}_e(x) \triangleq \mathbb{E}[c_e(1+X)] = \sum_{k=0}^{\infty} c_e(1+k) e^{-x} \frac{x^k}{k!}.$$

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2. *Along any convergent subsequence, the random resource-loads N_e^n converge in distribution to a Poisson random variable $N_e \sim \text{Poisson}(\tilde{x}_e)$, with $\tilde{x}_e$ the resource-loads in the corresponding Wardrop equilibrium $\tilde{y}$.*

Poisson convergence for vanishing probabilities

Corollary

If the costs $c_e : \mathbb{N} \to \mathbb{R}_+$ are monotone and non-constant, then $\tilde{c}_e(\cdot)$ are strictly monotone. Hence, the resource-loads $\tilde{x}_e$ are the same in any Wardrop equilibrium, and for every sequence π^n of Bayes-Nash equilibria we have

$$N_e^n \xrightarrow{\mathscr{D}} N_e \sim \mathit{Poisson}(\tilde{x}_e).$$

Summary and Comments

1. Both $w_i^n \to 0$ and $p_i^n \to 0$ lead to different non-atomic games in the limit.
   - For vanishing weights, the random resource-loads W_e^n converge in distribution to the constants resource-loads $\bar{x}_e$.
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4. However, Poisson games were defined without reference to a limit process, so the convergence result seems new. Also the connection between Poisson games and nonatomic games seems to be novel.

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Stochastic ACGs with Homogeneous Players

From now on we consider Stochastic ACGs with homogeneous players with unit weights $w_i \equiv 1$ and the same probabilities of being active $\mathbb{P}(Y_i = 1) \equiv p$.

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We are interested in how the Price-of-Anarchy varies as a function of p when we move from the deterministic case $p = 1$ to the limit when $p \downarrow 0$.

PoA for Bayes-Nash Equilibria

The expected cost for player i is

$$C_i^p(\pi) = p\,\mathbb{E}\left[\sum_{e\in E} X_{i,e} c_e^p(N_e)\right]$$

and the total social cost is

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$$\begin{aligned} \mathrm{PoA}(p) &= \sup_{\mathscr{G}^p} \max_{\pi\in\mathscr{E}(\mathscr{G}^p)} \frac{C^p(\pi)}{C^p(\pi^*)} && \text{(Price-of-Anarchy)} \\ \mathrm{PoS}(p) &= \sup_{\mathscr{G}^p} \min_{\pi\in\mathscr{E}(\mathscr{G}^p)} \frac{C^p(\pi)}{C^p(\pi^*)} && \text{(Price-of-Stability)} \end{aligned}$$

Equivalent Deterministic Game for Affine Costs

From now on we restrict to affine costs $c_e(x) = a_e + b_e\,x$ with $a_e, b_e \geq 0$. Hence

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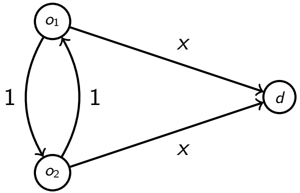
Example. Stochastic routing game with 2 homogeneous players



	R	U
R	$(1, 1)$	$(1+p, 2+p)$
D	$(2+p, 1+p)$	$(2, 2)$

Related Literature

- Related models
  - Non-atomic with stochastic demand (Wang, Doan and Chen, 2014; Correa, Hoeksma and Schröder, 2019)
  - Smoothness with incomplete information (Roughgarden, 2015)
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As a consequence of the latter we get $\mathrm{PoA}(p) \leq \frac{5}{2}$.

But we can find sharper bounds... and we expect $\mathrm{PoA}(p) \sim \frac{4}{3}$ for small p.

Smoothness Framework

Lemma (Roughgarden, 2015)

Let $\mathscr{G}$ be an unweighted atomic congestion game which is (λ, μ)-smooth with $\lambda > 0$ and $\mu \in (0, 1)$, that is to say

$$(\forall s, s' \in S) \quad \sum_{i \in N} C_i(s'_i, s_{-i}) \leq \lambda\, C(s') + \mu\, C(s).$$

Then we have $\mathrm{PoA}(\mathscr{G}) \leq \frac{\lambda}{1-\mu}$.

Smoothness Framework

Lemma

Let $\mathscr{P} = \{(k, m) \in \mathbb{N}^2 : k \geq 1\}$ *and suppose that* $\lambda > 0$ *and* $\mu \in (0, 1)$ *satisfy*

$$k\,(1+pm) \leq \lambda\, k\,(1-p+pk) + \mu\, m\,(1-p+pm) \qquad \forall (k, m) \in \mathscr{P}. \tag{1}$$

Then every stochastic ACG $\mathscr{G}^p$ *with homogeneous players and affine costs is* (λ, μ)*-smooth, and therefore* $\mathrm{PoA}(p) \leq \frac{\lambda}{1-\mu}$.

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Then every stochastic ACG $\mathscr{G}^p$ *with homogeneous players and affine costs is* (λ, μ)*-smooth, and therefore* $\mathrm{PoA}(p) \leq \frac{\lambda}{1-\mu}$.

The best combination of λ and μ for fixed p requires to solve

$$B(p) \triangleq \min_{\lambda>0, \mu\in(0,1)} \left\{ \frac{\lambda}{1-\mu} : \text{ subject to (1)} \right\}$$

which reduces to a 1D problem noting that the smallest λ compatible with (1) is

$$\lambda = \sup_{(k,m)\in\mathscr{P}} \frac{k(1+pm) - \mu\, m(1-p+pm)}{k(1-p+pk)}$$

Smoothness Framework

The previous reduction leads to the equivalent minimization problem

$$B(p) = \inf_{\mu \in (0,1)} \varphi_p\left(\tfrac{\mu}{1-\mu}\right) = \inf_{y>0} \varphi_p(y)$$

where $\varphi_p(\cdot)$ is the convex envelop function

$$\varphi_p(y) = \sup_{(k,m)\in\mathscr{P}} \tfrac{1+pm}{1-p+pk} + \tfrac{k(1+pm)-m(1-p+pm)}{k(1-p+pk)}\, y.$$

For each p the unique optimum y can be found explicitly, and then we recover the optimal combination (λ, μ).

Upper Bounds for the Price-of-Anarchy

Set $\bar{p}_0 = \frac{1}{4}$ and let $\bar{p}_1 \sim 0.3774$ be the unique real root of $8p^3 + 4p^2 = 1$.

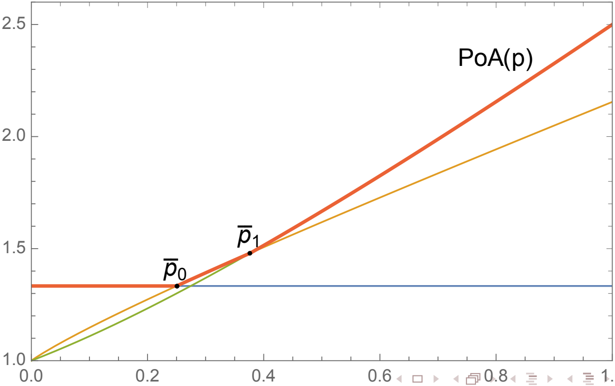
Theorem

The optimal solution for $B(p)$ is

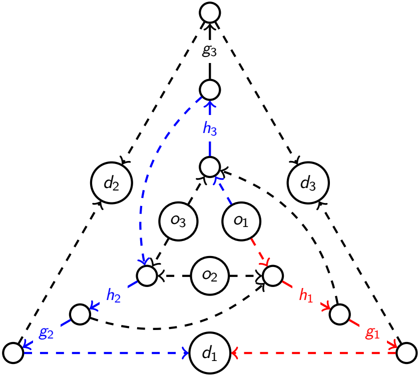
$$(\lambda, \mu) = \begin{cases} \left(1, \frac{1}{4}\right) & \text{if } 0 < p \leq \bar{p}_0, \\ \left(\frac{1+p+\sqrt{p(2+p)}}{2}, \frac{1+p-\sqrt{p(2+p)}}{2}\right) & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1, \\ \left(\frac{1+2p+2p^2}{1+2p}, \frac{p}{1+2p}\right) & \text{if } \bar{p}_1 \leq p \leq 1, \end{cases}$$

Upper Bounds for the Price-of-Anarchy

$$\mathrm{PoA}(p) \leq B(p) = \begin{cases} \frac{4}{3} & \text{if } 0 < p \leq \bar{p}_0, \\ \frac{1+p+\sqrt{p(2+p)}}{1-p+\sqrt{p(2+p)}} & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1, \\ 1+p+\frac{p^2}{1+p} & \text{if } \bar{p}_1 \leq p \leq 1, \end{cases}$$



Lower Bounds for Large p

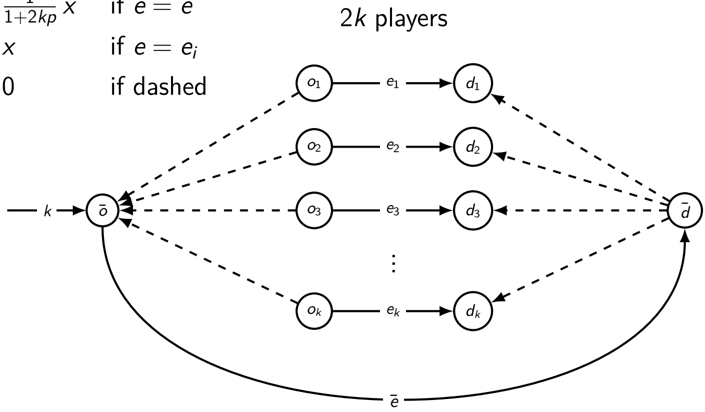


$$c_e(x) = \begin{cases} x & \text{if } e = h_i \\ p\,x & \text{if } e = g_i \\ 0 & \text{if dashed} \end{cases}$$

$$\Rightarrow \mathrm{PoA}(\mathscr{G}^p) = 1 + p + \frac{p^2}{1+p}.$$

Lower Bounds for Small p

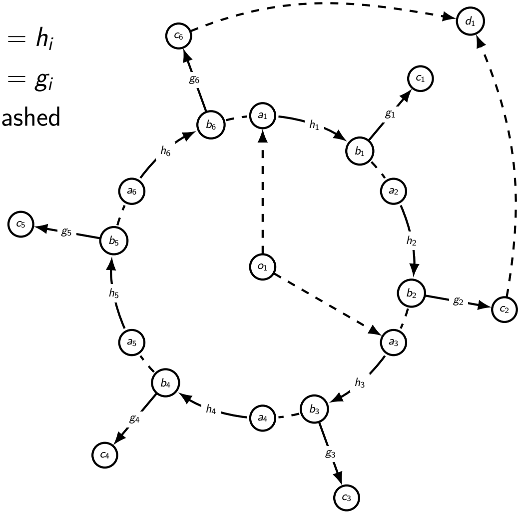
$$c_e(x) = \begin{cases} \frac{1}{1+2kp} x & \text{if } e = \bar{e} \\ x & \text{if } e = e_i \\ 0 & \text{if dashed} \end{cases}$$



$$\Rightarrow \mathrm{PoA}(\mathscr{G}^p) = \mathrm{PoS}(\mathscr{G}^p) \geq \frac{4kp+2-2p}{3kp+2-p} \to \frac{4}{3} \text{ as } k \to \infty$$

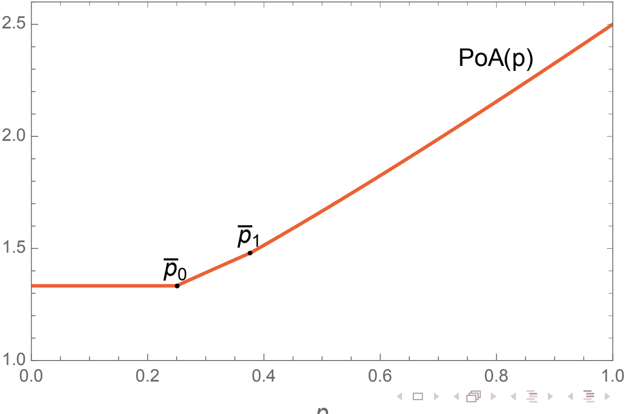
Lower Bounds for Intermediate p

$$c_e(x) = \begin{cases} \alpha\, x & \text{if } e = h_i \\ x & \text{if } e = g_i \\ 0 & \text{if dashed} \end{cases}$$



Bounds on the Price-of-Anarchy are Tight

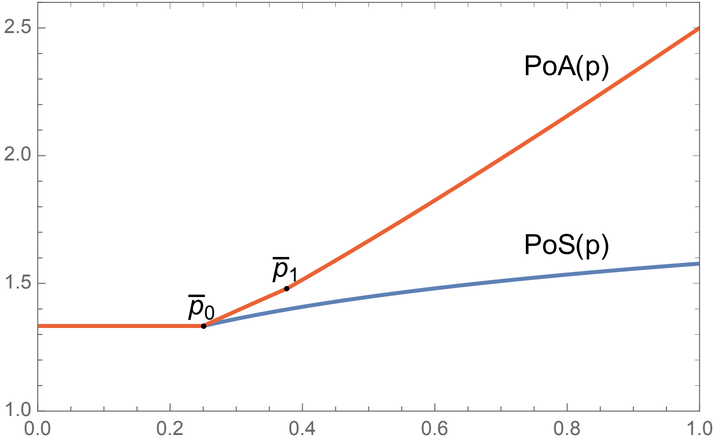
$$\mathrm{PoA}(p) = B(p) = \begin{cases} \frac{4}{3} & \text{if } 0 < p \leq \bar{p}_0 \\ \frac{1+p+\sqrt{p(2+p)}}{1-p+\sqrt{p(2+p)}} & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1 \\ 1+p+\frac{p^2}{1+p} & \text{if } \bar{p}_1 \leq p \leq 1 \end{cases}$$



Price-of-Anarchy vs Price-of-Stability

Combining with Kleer and Schäfer (2018), we also get tight bounds for PoS

$$\mathrm{PoS}(p) = \begin{cases} 4/3 & \text{if } 0 < p \leq \bar{p}_0 \\ 1 + \sqrt{p/(2+p)} & \text{if } p \geq \bar{p}_0 \end{cases}$$



Conclusion

1. Convergence towards non-atomic games:
   - vanishing weights $\longrightarrow$ Wardrop
   - vanishing probabilities $\longrightarrow$ Poisson
2. Tight bounds on PoA/PoS for affine costs

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1. Convergence towards non-atomic games:
   - vanishing weights $\longrightarrow$ Wardrop
   - vanishing probabilities $\longrightarrow$ Poisson
2. Tight bounds on PoA/PoS for affine costs
3. Some open questions
   - Mixed limits: weights & probabilities
   - Bounds on PoA for heterogeneous p_i's
   - Tight bounds for general costs: quadratic, polynomial,...
   - Continuity of PoA/PoS:

$$\mathrm{PoA}(w^n) \xrightarrow{?} \mathrm{PoA}(\text{Wardrop})$$

$$\mathrm{PoA}(p^n) \xrightarrow{?} \mathrm{PoA}(\text{Poisson})$$

   - Stronger notion of optimal: prophet vs non-prophet

Questions ?