Atomic Congestion Games with Stochastic Demand
Convergence and Price-of-Anarchy

R. Cominetti\textsuperscript{1}, M. Scarsini\textsuperscript{2}, M. Schröder\textsuperscript{3}, N. Stier-Moses\textsuperscript{4}

\textsuperscript{1}Universidad Adolfo Ibáñez
\textsuperscript{2}LUISS
\textsuperscript{3}RWTH Aachen
\textsuperscript{4}Facebook

Network, Population and Congestion Games
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You are planning your commute route for tomorrow.

Not sure about your departure time, nor who might be on the road.

A game with a random set of players!
Outline – Convergence

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Introduction

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Can we precise in which sense the discrete and continuous models are close?
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Games involving "many small players" are conveniently modeled as nonatomic games with a continuum of players.

Can we precise in which sense the discrete and continuous models are close?

This depends on what we mean by "small"... For instance:

- Player $i$ has a small load $w_i \sim 0$ to be transported with certainty,
- Player $i$ has a unit load but is present with small probability $p_i \sim 0$.

Depending on which limit we consider, we get different nonatomic models.
Outline – Price of Anarchy

Focusing on atomic congestion games with *affine costs and stochastic demands* where each player is present with probability $p$, we study how

$$\text{PoA}(p) = \frac{\text{Social cost of worst equilibrium}}{\text{Least possible social cost}}$$

as a function of $p$. 

![Graph showing PoA(p) and PoS(p) as functions of p](graph.png)
Introduction

Convergence of Congestion Games

- Nonatomic games and Wardrop equilibria
- Weighted atomic games: convergence of Nash equilibria
- Stochastic atomic games: convergence of Bayes-Nash equilibria

Price-of-Anarchy for Stochastic ACGs with Affine Costs

- Upper bounds
- Lower bounds
- Price-of-Stability
A *congestion game* is described by

- a set of *resources* $e \in E$ with continuous monotone costs $c_e : \mathbb{R} \rightarrow \mathbb{R}_+$
- a set of *types* $t \in T$ with corresponding strategy sets $S_t \subseteq 2^E$
- a (continuous or discrete) *demand* $d_t \geq 0$ for each type $t \in T$
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**Example:** In routing games the resources are the edges of a network \( G = (V, E) \), \( T \) is the set of OD pairs, and \( S_t \) the paths connecting \( (o_t, d_t) \).
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**Non-atomic:** continuous, fine grained, many players \( \rightarrow \) urban traffic

**Atomic splittable:** continuous, few players \( \rightarrow \) fluids, sand, telecom
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![Diagram](image)

**Non-atomic:** continuous, fine grained, many players $\rightarrow$ urban traffic

**Atomic splittable:** continuous, few players $\rightarrow$ fluids, sand, telecom

**Atomic unsplittable:** discrete, few players $\rightarrow$ vessels, airplanes

**Stochastic:** unpredictable $\rightarrow$ packets or vehicles over a network
Non-Atomic Congestion Games

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A Wardrop equilibrium is a decomposition of the demands $d_t = \sum_{s \in S_t} y_s$ into strategy flows $y_s \geq 0$ such that only minimum cost strategies are used, i.e.:

$$\sum_{e \in E} c_e(x_e) \geq \sum_{e \in E} c_e(x_{e}')$$
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$$(\forall t \in T)(\forall s, s' \in S_t) \quad y_s > 0 \Rightarrow \sum_{e \in s} c_e(x_e) \leq \sum_{e \in s'} c_e(x_e)$$
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where \(x_e = \sum_{s \ni e} y_s\) are the induced resource-loads.
Atomic Splittable Congestion Games

Atomic splittable congestion games are similar to non-atomic models in that demands are continuous and can be split arbitrarily over different strategies.

The fundamental differences are:

1. There are finitely many players, each one controls a fraction of the demand.
2. Each player has a non-negligible effect on congestion and exploits her *market power* by strategically splitting the demand over the available strategies.
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**Theorem (Haurie & Marcotte, 1985)**

*When the number of players increases and the demand controlled by each of them tends to 0, the splittable equilibria converge to a Wardrop equilibrium.*
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For the precise statement and subsequent generalizations, see Jacquot & Wang (2018) and references therein.

Here we address the discrete cases: unsplittable and stochastic demands.
Weighted Atomic Congestion Games

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- For a strategy profile $s = (s_i)_{i \in N}$ with $s_i \in S_{t_i}$ we denote $X_{i,e} = \mathbb{1}_{\{e \in s_i\}}$.
- The corresponding resource-loads are $W_e = \sum_{i \in N} w_i X_{i,e}$.
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Given mixed strategies $\pi_i \in \Delta(S_{t_i})$, the Bernoulli random variables $X_{i,e}$ are independent across players with $\mathbb{P}(X_{i,e} = 1) = \sum_{s_i \ni e} \pi_i(s_i)$. 


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A mixed strategy profile \( \pi = (\pi_i)_{i \in N} \) is a **Nash equilibrium** iff for each player \( i \) and strategies \( s, s' \in S_{t_i} \) with \( \pi_i(s) > 0 \) we have

\[
\sum_{e \in s} \mathbb{E}[c_e(W_e) | X_{i,e} = 1] \leq \sum_{e \in s'} \mathbb{E}[c_e(W_e) | X_{i,e} = 1]
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- ACGs with identical weights $w_i \equiv \bar{w}$ are potential games and admit pure equilibria (Rosenthal’73).
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**Example.** A 2-player routing game with equal weights \( w_i = 1 \)

\[
\begin{array}{c|cc}
& R & U \\
\hline
R & (1, 1) & (2, 3) \\
D & (3, 2) & (2, 2) \\
\end{array}
\]
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```
\begin{align*}
\text{Example: } \text{Routing 10 players over 2 identical parallel links.}
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```
Wardrop Convergence for Vanishing Weights

**Theorem**

Consider an arbitrary sequence $\pi^n$ of mixed equilibria for a sequence of weighted ACGs with player sets $N = \{1, \ldots, n\}$ and weights $w^n_i$ such that

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\begin{align*}
\text{a)} \quad & \max_{i \in N} w^n_i \to 0 \\
\text{b)} \quad & (\forall t \in T) \quad d^n_t \triangleq \sum_{i: t^n_i = t} w^n_i \to d_t
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Then

1. The sequence $y^n$ of expected strategy loads $y^n_s = \sum_i w^n_i \pi^n_i(s)$ is bounded and each accumulation point $\bar{y}$ is a Wardrop equilibrium for the nonatomic game with demands $d_t$ and costs $c_e(\cdot)$. 

(Institut Henri Poincaré)
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2. Along any convergent subsequence, the random resource-loads $W^n_e$ converge in distribution to the constant resource-loads $\bar{x}_e$ in the Wardrop equilibrium $\bar{y}$. 
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**Remark:** If the $c_e(\cdot)$’s are strictly monotone, then $\bar{x}$ is unique and $W^n_e \xrightarrow{D} \bar{x}_e$. 

(Institut Henri Poincaré)
Stochastic Atomic Congestion Games

A *stochastic atomic congestion game* features finitely many players $i \in N$ with types $t_i \in T$, unit weights $w_i = 1$, and a probability of being active

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A **strategy profile** $\pi = (\pi_i)_{i \in N}$ is a **Bayes-Nash equilibrium** if for each player $i$ and strategies $s, s' \in S_{t_i}$ with $\pi_i(s) > 0$ we have

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Remark. The costs $c_e(N_e)$ need only be defined over the integers $\mathbb{N}$, and the continuity assumption becomes irrelevant.
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**Remark.** The costs $c_e(\cdot)$ need only be defined over the integers $c_e : \mathbb{N} \to \mathbb{R}_+$, and the continuity assumption becomes irrelevant.
Stochastic ACGs are Potential Games

**Theorem**

*Every stochastic ACG is a potential game, hence it has pure Nash equilibria, with potential given by*

\[
\Phi(s) \triangleq \mathbb{E} \left[ \sum_{e \in E} \sum_{k=1}^{N_e(s)} c_e(k) \right]
\]

*where* \(N_e(s) = \sum_{i \in N} Y_i \mathbb{1}_{\{e \in s_i\}}\).*
Poisson Convergence for Vanishing Probabilities

Theorem

Let $\pi^n$ be an arbitrary sequence of Bayes-Nash equilibria for a sequence of stochastic ACGs with unit weights $w_i = 1$ and probabilities $p^n_i$ such that

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Suppose further that $\mathbb{E}[X^2 c_e(1+X)] < \infty$ for every $X \sim \text{Poisson}(x)$, and set

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\tilde{c}_e(x) \triangleq \mathbb{E}[c_e(1+X)] = \sum_{k=0}^{\infty} c_e(1+k) e^{-x} \frac{x^k}{k!}.
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1. The sequence $y^n$ of expected strategy loads $y^n_s = \sum_i p_i^n \pi_i^n(s)$ is bounded and each accumulation point $\tilde{y}$ is a Wardrop equilibrium for the non-atomic congestion game with demands $d_t$ and costs $\tilde{c}_e(\cdot)$. 
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2. Along any convergent subsequence, the random resource-loads $N^n_e$ converge in distribution to a Poisson random variable $N_e \sim \text{Poisson}(\tilde{x}_e)$, with $\tilde{x}_e$ the resource-loads in the corresponding Wardrop equilibrium $\tilde{y}$.
Poisson convergence for vanishing probabilities

Corollary

If the costs $c_e : \mathbb{N} \rightarrow \mathbb{R}_+$ are monotone and non-constant, then $\tilde{c}_e(\cdot)$ are strictly monotone. Hence, the resource-loads $\tilde{x}_e$ are the same in any Wardrop equilibrium, and for every sequence $\pi^n$ of Bayes-Nash equilibria we have

$$N^n_e \overset{D}{\rightarrow} N_e \sim \text{Poisson}(\tilde{x}_e).$$
Summary and Comments

1. Both $w_i^n \to 0$ and $p_i^n \to 0$ lead to different non-atomic games in the limit.

   - For vanishing weights, the random resource-loads $W^n_e$ converge in distribution to the constants resource-loads $\bar{x}_e$.
   - For vanishing probabilities, $N^n_e$ remain random in the limit and converge in distribution to some $N_e \sim \text{Poisson}(\bar{x}_e)$.
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2. The latter seems more appropriate to capture the randomness observed in real networks. Also $p^n_i \to 0$ is quite natural... congestion depends on players that are present on a small window around your departure time.
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3. The Poisson limit can be shown to be a special case of Myerson’s Poisson games (Int J Game Theory 1998): the normalized limit flows $\sigma(s|t) = \tilde{y}_s/d_t$ for $s \in S_t$ are in fact an equilibrium in the Poisson game.
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4. However, Poisson games were defined without reference to a limit process, so the convergence result seems new. Also the connection between Poisson games and nonatomic games seems to be novel.
1. Introduction

2. Convergence of Congestion Games
   - Nonatomic games and Wardrop equilibria
   - Weighted atomic games: convergence of Nash equilibria
   - Stochastic atomic games: convergence of Bayes-Nash equilibria

3. Price-of-Anarchy for Stochastic ACGs with Affine Costs
   - Upper bounds
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   - Price-of-Stability
Stochastic ACGs with Homogeneous Players

From now on we consider Stochastic ACGs with homogeneous players with unit weights $w_i \equiv 1$ and the same probabilities of being active $\mathbb{P}(Y_i = 1) \equiv p$. 
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**Proposition**

A Stochastic ACG with homogeneous players is equivalent to a deterministic unweighted ACG for the auxiliary costs

$$c_e^p(k) = \mathbb{E}[c_e(1 + B)] \text{ with } B \sim \text{Binomial}(k-1, p)$$
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A Stochastic ACG with homogeneous players is equivalent to a deterministic unweighted ACG for the auxiliary costs

\[
c_e^p(k) = \mathbb{E}[c_e(1 + B)] \text{ with } B \sim \text{Binomial}(k-1, p)
\]

We are interested in how the Price-of-Anarchy varies as a function of \( p \) when we move from the deterministic case \( p = 1 \) to the limit when \( p \downarrow 0 \).
PoA for Bayes-Nash Equilibria

The expected cost for player $i$ is

$$C^p_i(\pi) = p \mathbb{E} \left[ \sum_{e \in E} X_{i,e} c_e^p(N_e) \right]$$

and the total social cost is

$$C^p(\pi) = \sum_{i \in N} C^p_i(\pi) = p \mathbb{E} \left[ \sum_{e \in E} N_e c_e^p(N_e) \right].$$

A strategy profile $\pi^*$ minimizing $C^p(\cdot)$ is called a social optimum.
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A strategy profile $\pi^*$ minimizing $C^p(\cdot)$ is called a social optimum.

$$\text{PoA}(p) = \sup_{\mathcal{G}_p} \max_{\pi \in \mathcal{E}(\mathcal{G}_p)} \frac{C^p(\pi)}{C^p(\pi^*)} \quad \text{(Price-of-Anarchy)}$$

$$\text{PoS}(p) = \sup_{\mathcal{G}_p} \min_{\pi \in \mathcal{E}(\mathcal{G}_p)} \frac{C^p(\pi)}{C^p(\pi^*)} \quad \text{(Price-of-Stability)}$$
Equivalent Deterministic Game for Affine Costs

From now on we restrict to affine costs \( c_e(x) = a_e + b_e x \) with \( a_e, b_e \geq 0 \). Hence

\[
c^p_e(k) = \mathbb{E}[c_e(1 + B(k-1, p))] \\
= a_e + b_e(1 + (k-1)p) \\
= a^p_e + b^p_e k
\]
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\]

**Example.** Stochastic routing game with 2 homogeneous players

\[
\begin{array}{c|cc}
 & R & U \\
\hline
R & (1, 1) & (1+p, 2+p) \\
D & (2+p, 1+p) & (2, 2) \\
\end{array}
\]
Related Literature

- Related models
  - Non-atomic with stochastic demand (Wang, Doan and Chen, 2014; Correa, Hoeksma and Schröder, 2019)
  - Smoothness with incomplete information (Roughgarden, 2015)
  - Perception based (Kleer and Schäfer, 2018)
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- PoA for congestion games with affine costs
  - $\text{PoA}(\mathcal{G}) \leq \frac{4}{3}$ for non-atomic (Roughgarden and Tardos, 2002)
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As a consequence of the latter we get \( \text{PoA}(p) \leq \frac{5}{2} \).

But we can find sharper bounds... and we expect \( \text{PoA}(p) \sim \frac{4}{3} \) for small \( p \).
Smoothness Framework

Lemma (Roughgarden, 2015)

Let $G$ be an unweighted atomic congestion game which is $(\lambda, \mu)$-smooth with $\lambda > 0$ and $\mu \in (0, 1)$, that is to say

$$(\forall s, s' \in S) \sum_{i \in N} C_i(s'_i, s_{-i}) \leq \lambda C(s') + \mu C(s).$$

Then we have $\text{PoA}(G) \leq \frac{\lambda}{1-\mu}.$
Smoothness Framework

Lemma

Let $\mathcal{P} = \{(k, m) \in \mathbb{N}^2 : k \geq 1\}$ and suppose that $\lambda > 0$ and $\mu \in (0, 1)$ satisfy

\[ k(1+pm) \leq \lambda k(1-p+pk) + \mu m(1-p+pm) \quad \forall (k, m) \in \mathcal{P}. \tag{1} \]

Then every stochastic ACG $\mathcal{G}^p$ with homogeneous players and affine costs is $(\lambda, \mu)$-smooth, and therefore $\text{PoA}(p) \leq \frac{\lambda}{1-\mu}$. 
Smoothness Framework

Lemma

Let $\mathcal{P} = \{(k, m) \in \mathbb{N}^2 : k \geq 1\}$ and suppose that $\lambda > 0$ and $\mu \in (0, 1)$ satisfy

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Then every stochastic ACG $\mathcal{G}^p$ with homogeneous players and affine costs is $(\lambda, \mu)$-smooth, and therefore $\text{PoA}(p) \leq \frac{\lambda}{1-\mu}$.

The best combination of $\lambda$ and $\mu$ for fixed $p$ requires to solve

$$B(p) \triangleq \min_{\lambda > 0, \mu \in (0,1)} \left\{ \frac{\lambda}{1-\mu} : \text{subject to (1)} \right\}$$

which reduces to a 1D problem noting that the smallest $\lambda$ compatible with (1) is

$$\lambda = \sup_{(k,m) \in \mathcal{P}} \frac{k(1+pm)-\mu m(1-p+pm)}{k(1-p+pk)}$$
Smoothness Framework

The previous reduction leads to the equivalent minimization problem

\[ B(p) = \inf_{\mu \in (0,1)} \varphi_p \left( \frac{\mu}{1-\mu} \right) = \inf_{y > 0} \varphi_p(y) \]

where \( \varphi_p(\cdot) \) is the convex envelope function

\[ \varphi_p(y) = \sup_{(k,m) \in \mathcal{D}} \frac{1+pm}{1-p+pk} + \frac{k(1+pm)-m(1-p+pm)}{k(1-p+pk)} y. \]

For each \( p \) the unique optimum \( y \) can be found explicitly, and then we recover the optimal combination \((\lambda, \mu)\).
Set \( \bar{p}_0 = \frac{1}{4} \) and let \( \bar{p}_1 \sim 0.3774 \) be the unique real root of \( 8p^3 + 4p^2 = 1 \).

**Theorem**

The optimal solution for \( B(p) \) is

\[
(\lambda, \mu) = \begin{cases} 
(1, \frac{1}{4}) & \text{if } 0 < p \leq \bar{p}_0, \\
\left( \frac{1+p+\sqrt{p(2+p)}}{2}, \frac{1+p-\sqrt{p(2+p)}}{2} \right) & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1, \\
\left( \frac{1+2p+2p^2}{1+2p}, \frac{p}{1+2p} \right) & \text{if } \bar{p}_1 \leq p \leq 1, 
\end{cases}
\]
Upper Bounds for the Price-of-Anarchy

\[
\text{PoA}(p) \leq B(p) = \begin{cases} 
  \frac{4}{3} & \text{if } 0 < p \leq \bar{p}_0, \\
  \frac{1+p+\sqrt{p(2+p)}}{1-p+\sqrt{p(2+p)}} & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1, \\
  1 + p + \frac{p^2}{1+p} & \text{if } \bar{p}_1 \leq p \leq 1,
\end{cases}
\]

Graph showing the function PoA(p) with points \( \bar{p}_0 \) and \( \bar{p}_1 \) marked on the graph.
Lower Bounds for Large $p$

$$c_e(x) = \begin{cases} x & \text{if } e = h_i \\ px & \text{if } e = g_i \\ 0 & \text{if dashed} \end{cases}$$

$$\Rightarrow \text{PoA}(G^p) = 1 + p + \frac{p^2}{1 + p}.$$
Lower Bounds for Small $p$

$$c_e(x) = \begin{cases} 
\frac{1}{1+2kp} x & \text{if } e = \bar{e} \\
 x & \text{if } e = e_i \\
0 & \text{if dashed}
\end{cases}$$

$$\Rightarrow \text{PoA}(\mathcal{G}) = \text{PoS}(\mathcal{G}) \geq \frac{4kp+2-2p}{3kp+2-p} \to \frac{4}{3} \text{ as } k \to \infty$$
Lower Bounds for Intermediate $p$

\[ c_e(x) = \begin{cases} 
\alpha x & \text{if } e = h_i \\
x & \text{if } e = g_i \\n0 & \text{if dashed}
\end{cases} \]
Bounds on the Price-of-Anarchy are Tight

\[
\text{PoA}(p) = B(p) = \begin{cases} 
\frac{4}{3} & \text{if } 0 < p \leq \bar{p}_0 \\
\frac{1+p+\sqrt{p(2+p)}}{1-p+\sqrt{p(2+p)}} & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1 \\
1 + p + \frac{p^2}{1+p} & \text{if } \bar{p}_1 \leq p \leq 1 
\end{cases}
\]
Price-of-Anarchy vs Price-of-Stability

Combining with Kleer and Schäfer (2018), we also get tight bounds for PoS

\[ \text{PoS}(p) = \begin{cases} 
4/3 & \text{if } 0 < p \leq \bar{p}_0 \\
1 + \sqrt{p/(2 + p)} & \text{if } p \geq \bar{p}_0
\end{cases} \]

\[ \text{PoA}(p) \]

\[ \text{PoS}(p) \]
Conclusion

1. Convergence towards non-atomic games:
   - vanishing weights $\rightarrow$ Wardrop
   - vanishing probabilities $\rightarrow$ Poisson

2. Tight bounds on PoA/PoS for affine costs
Conclusion

1. Convergence towards non-atomic games:
   - vanishing weights $\rightarrow$ Wardrop
   - vanishing probabilities $\rightarrow$ Poisson

2. Tight bounds on PoA/PoS for affine costs

3. Some open questions
   - Mixed limits: weights & probabilities
   - Bounds on PoA for heterogeneous $p_i$'s
   - Tight bounds for general costs: quadratic, polynomial,...
   - Continuity of PoA/PoS:
     \[
     \begin{align*}
     \text{PoA}(w^n) & \xrightarrow{?} \text{PoA}(\text{Wardrop}) \\
     \text{PoA}(p^n) & \xrightarrow{?} \text{PoA}(\text{Poisson})
     \end{align*}
     \]
   - Stronger notion of optimal: prophet vs non-prophet
Questions?