Mean field approximation for (relatively) small populations

Nicolas Gast

Inria, Grenoble, France (joint work with Benny Van Houdt (Univ. Antwerp))

Workshop on Network, population and congestion games, April 2019
Markov models do not scale

- An agent evolves in a finite state-space: $S(t) \in S$. The system is described by a Kolmogorov equation:

$$\frac{d}{dt} P[S(t) = s] = \sum_{s'} P[S(t) = s'] Q_{s',s}.$$

Works well if $|S|$ is “small”
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\]

Works well if \(|S|\) is “small”

Problem with population: state space explosion
\( S \) states per agent, \( N \) agents \( \Rightarrow S^N \) states

Main problem: correlations

\[
P[A, B] \neq P[A] P[B]
\]
Solution: Mean field approximation, Propagation of Chaos

When a population becomes “large”:

(mean field) a single agent has a minor influence on the mass.

\[
S_1 \perp \frac{1}{N} \sum_{n=1}^{N} \delta_{S_n} \quad \text{(as } N \to \infty)\]

(prop. of chaos) Any finite subset of objects become independent:

\[
P [S_1, \ldots S_k] \approx P [S_1] \ldots P [S_k] \quad \text{(as } N \to \infty)\]
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(\text{mean field}) \quad \text{a single agent has a minor influence on the mass.}

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(\text{prop. of chaos}) \quad \text{Any finite subset of objects become independent:}

\[
P [S_1, \ldots S_k] \approx P [S_1] \ldots P [S_k] \quad \text{(as } N \to \infty)\]

\text{Good. Reduces the complexity from } S^N \text{ equations to } SN!\]

\text{Bad. Why should this be OK? (or when?)}
Discrete space mean field model

Population of $N$ agents
- Each agent evolves in a finite state-space $S_n(t) \in S$. 

Mean Field Interaction Model
- Evolution of one agent: Markov kernel $Q(X)$.
- $X_i$ = fraction of agents in state $i$.
- $Q_{ij}(X)$ = rate/proba of one agent jumping from $i$ to $j$.
- $Q(\cdot)$ is supposed given and can represent:
  - Replicator dynamics
  - Best-response dynamics
  - Effect of environment
  - Result of centralized/decentralized optimization
Discrete space mean field model

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**Mean Field Interaction Model**

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$Q(.)$ is supposed *given* and can represent:

- Replicator dynamic, Best-response dynamics
- Effect of environment
- Result of centralized/decentralized optimization
Mean field approximation

When the number of agents is large, agents become independent:

- In the synchronous case\(^1\):
  \[ X(t + 1) = X(t)Q(X(t)) \]

- In the asynchronous case\(^2\).
  \[ \frac{d}{dt}X(t) = X(t)Q(X(t)) \]

In this talk, I will focus on the latter.

---

\(^1\) Gomes, Mohr, Souza, 2010: Discrete time, finite state space mean field games

\(^2\) Gomes, Mohr, Souza 2013: Continuous time finite state mean field game
This talk: relation between finite $N$ models and mean field approximation

\[
\lim_{N \to \infty} \begin{pmatrix}
0 & 2 & 4 \\
0.0 & 0.1 & 0.2 \\
N = 100 & & \\
\end{pmatrix}
= 
\begin{pmatrix}
0 & 2 & 4 \\
0.0 & 0.1 & 0.2 \\
\text{ODE (}N = \infty\text{)} & & \\
\end{pmatrix}
\]

Mean field approximation $\dot{x} = xQ(x)$

\[
P[S_n(t) = i] \approx X_i(t) \approx x_i(t).
\]
Some examples

Information propagation
\( x = \text{fraction of "informed" people} \)

Outdated \( \xrightarrow{1} \) Informed

Load balancing (supermarket model)
(Mitzenmacher 98, Vvedenskaya 96)

Cache
G., Van Houdt 2015

802.11 (wireless)
Bianchi 2000, Le Boudec, Cho 2011
Outline

1. Population Processes

2. Moment closure and refined mean field approximation

3. Conclusion : Does it always work?
Before studying a generic model, let us look at a simple example

Transitions:

\[ X \mapsto X + \frac{1}{N} \quad \text{rate } N(1 - X)(1 + X) \]
\[ X \mapsto X - \frac{1}{N} \quad \text{rate } NX \]
Before studying a generic model, let us look at a simple example

Transitions:

\[ X \leftrightarrow X + \frac{1}{N} \quad \text{rate } N(1 - X)(1 + X) \]
\[ X \leftrightarrow X - \frac{1}{N} \quad \text{rate } NX \]

Drift:

\[
\frac{d}{dt} \mathbb{E}[X(t)] = \mathbb{E} \left[ \frac{1}{N} N(1 - X)(1 + X) - \frac{1}{N} NX \right] = \mathbb{E} \left[ 1 - X - X^2 \right]
\]

Mean field approximation:

\[
\dot{x} = 1 - x - x^2
\]
We study a population of $N$ interchangeable agents where $O(1)$ agents change states at the same time.

$X$ denotes the empirical measure.

$$X_i(t) = \text{fraction of agents in state } i$$

Transitions are:

$$X \mapsto X + \frac{\ell}{N} \quad \text{at rate } N\beta_\ell(X).$$

The mean field-approximation is the solution of $\dot{x} = f(x)$ where

$$f(x) = \sum_\ell \ell \beta_\ell(x)$$

$^3(E, \|\cdot\|)$ is a subset of a Banach space, typically $\mathbb{R}^d$. 
Population processes become deterministic as $N \to \infty$

Theorem (Kurtz (1970s), Ying (2016)):

If the drift $f$ is Lipschitz-continuous:

$$X^N(t) \approx x(t) + \frac{1}{\sqrt{N}} G_t$$

If in addition the ODE has a unique attractor $\pi$:

$$\mathbb{E} \left[ X^N(\infty) - \pi \right] = O(1/\sqrt{N})$$
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Take-home message

For a population process with homogeneous interactions:
- The mean field approximation is asymptotically exact
  - Functional law of large number
- The population $X$ is at distance $1/\sqrt{N}$ from the mean field.
  - Functional central limit theorem
Outline

1. Population Processes

2. Moment closure and *refined* mean field approximation

3. Conclusion: Does it always work?
What changes when one focus on performance evaluation?

Simulations results ($\rho = 0.9$)

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<tr>
<th>$N$</th>
<th>10</th>
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Theorem (Kolokoltsov 2012, G. 2017 & 2018). If the drift $f$ is $C^2$ and has a unique exponentially stable attractor, then for any $t \in [0, \infty) \cup \{\infty\}$, there exists a constant $V_t$ such that:

$$E[h(X_N(t))] = h(x(t)) + V_t N + O(1/N^2)$$
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Error seems to decrease as $1/N$

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$$E[h(X_N(t))]=h(x(t))+V_t N + O(1/N^2)$$

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Randomly choose two, and select one
What changes when one focuses on performance evaluation?

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$$\mathbb{E} \left[ h(X^N(t)) \right] = h(x(t)) + \frac{V(t)}{N} + O(1/N^2)$$
Where does the $1/N$-term comes from?

The moment closure approach

Going back to the information propagation example (and writing $X$ instead of $X(t)$, we get:

$$
\frac{d}{dt} \mathbb{E} [X] = \mathbb{E} \left[ 1 - X - X^2 \right]
$$

$$
= 1 - \mathbb{E} [X] - \mathbb{E} [X^2]
$$

Problem: this equation is not closed because we need $\mathbb{E} [X^2]$. 

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\frac{d}{dt} \mathbb{E}[X] = \mathbb{E}[1 - X - X^2] = 1 - \mathbb{E}[X] - \mathbb{E}[X^2]
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Problem: this equation is not closed because we need $\mathbb{E}[X^2]$.

Hence, there are two choices:

1. Assume $\mathbb{E}[X^2] \approx \mathbb{E}[X]^2$. This gives the mean field approximation:
   $$
   \dot{x} = 1 - x - x^2.
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1. Assume $\mathbb{E}[X^2] \approx \mathbb{E}[X]^2$. This gives the mean field approximation:
   \[\dot{x} = 1 - x - x^2.\]

2. Obtain an equation for $\mathbb{E}[X^2]$.

   \[
   X^2 \mapsto (X + \frac{1}{N})^2 \text{ at rate } N(1 - X^2) \\
   X^2 \mapsto (X - \frac{1}{N})^2 \text{ at rate } X
   \]
The moment closure approach (continued)

Hence on average:
\[
\frac{d}{dt} \mathbb{E} [X^2] = \mathbb{E} \left[ \left( \frac{2X}{N} + \frac{1}{N^2} \right) N (1 - X^2) + \left( -\frac{2X}{N} + \frac{1}{N^2} \right) N X \right] \\
= \mathbb{E} \left[ 2X - 2X^3 - 2X^2 + \frac{1}{N} (1 - X^2 + X) \right] \\
= 2\mathbb{E} [X] - 2\mathbb{E} [X^3] - 2\mathbb{E} [X^2] + \frac{1}{N} (1 - \mathbb{E} [X^2] + \mathbb{E} [X])
\]

Problem: this equation is not closed because we need \( \mathbb{E} [X^3] \).
The moment closure approach (continued)

Hence on average:

\[
\frac{d}{dt} \mathbb{E}[X^2] = \mathbb{E}\left[ \left( \frac{2X}{N} + \frac{1}{N^2} \right)N(1 - X^2) + \left( -\frac{2X}{N} + \frac{1}{N^2} \right)NX \right]
\]

\[
= \mathbb{E}\left[ 2X - 2X^3 - 2X^2 + \frac{1}{N}(1 - X^2 + X) \right]
\]

\[
= 2\mathbb{E}[X] - 2\mathbb{E}[X^3] - 2\mathbb{E}[X^2] + \frac{1}{N}(1 - \mathbb{E}[X^2] + \mathbb{E}[X])
\]

Problem: this equation is not closed because we need \( \mathbb{E}[X^3] \).

Hence, there are two choices:

1. Assume \( \mathbb{E}[X^3] \approx 3\mathbb{E}[X^2] \mathbb{E}[X] - 2\mathbb{E}[X]^3 \). This gives the second order moment closure approximation:

\[
\dot{x} = 1 - x - y
\]

\[
\dot{y} = 2x - (3xy - 2x^3) - 2y + \frac{1}{N}(1 - y + x)
\]

2. Obtain an equation for \( \mathbb{E}[X^3] \) (that will involve \( \mathbb{E}[X^4] \)...)}
Using this approach, we can derive $1/N^k$-expansions

**Theorem.** Assume that $f$ is $C^2$ and let $x$ be the solution of $\frac{d}{dt}x = f(x)$.

$$\frac{d}{dt} \mathbb{E}[X(t)] = x(t) + O(1/N).$$
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**Theorem.** Assume that $f$ is $C^2$ and let $x$ be the solution of $\frac{d}{dt} x = f(x)$.

\[
\frac{d}{dt} \mathbb{E}[X(t)] = x(t) + \frac{1}{N} V(t) + O(1/N^2).
\]

Let $Y(t) = X(t) - x(t)$. Then:

\[
\mathbb{E}[Y(t)] = \frac{1}{N} V(t) + O(1/N^2)
\]
\[
\mathbb{E}[Y(t) \otimes Y(t)] = \frac{1}{N} W(t) + O(1/N^2)
\]

where

\[
\frac{d}{dt} V^i = f^i_j V^j + f^i_{j,k} W^{j,k}
\]
\[
\frac{d}{dt} W^{j,k} = f^j_\ell W^{\ell,k} + f^k_\ell W^{j,\ell}
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Using this approach, we can derive $1/N^k$-expansions

**Theorem.** Assume that $f$ is $C^2$ and let $x$ be the solution of $\frac{d}{dt}x = f(x)$.

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$$\mathbb{E}[Y(t)] = \frac{1}{N} V(t) + \frac{1}{N^2} A(t) + O(1/N^3)$$

$$\mathbb{E}[Y(t) \otimes Y(t)] = \frac{1}{N} W(t) + \frac{1}{N^2} B(t) + O(1/N^3)$$

$$\text{esp} Y(t) \otimes^3 = \frac{1}{N^2} C(t) + O(1/N^3)$$

$$\text{esp} Y(t) \otimes^4 = \frac{1}{N^2} D(t) + O(1/N^3)$$

where:

\[
\begin{align*}
\frac{d}{dt} V^i &= f^i_j V^j + f^i_{j,k} W^j^k \\
\frac{d}{dt} W^{j,k} &= f^j_i W^i_k + f^j_{i,\ell} W^i_{\ell,k} \\
\frac{d}{dt} A^i &= f^i_j A^j + f^i_{j,k} B^j^k + f^i_{j,k,\ell} C^j^k^\ell + f^i_{j,k,\ell,m} D^j^k^\ell^m \\
\frac{d}{dt} B^{i,j} &= f^i_k B^{k,j} + f^j_k B^{i,k} + \frac{3}{2} \left[ f^i_k C^{k,j} + f^j_k C^{k,i} \right] + 2(f^i_{k,\ell} D^{k,\ell,j} + f^j_{k,\ell} D^{k,i}) + \frac{1}{2} Q^i_{k,j} V^k + \frac{1}{2} Q^i_{k,j} W^{k,\ell} 
\end{align*}
\]
Computational issues

Recall that \( x(t) \) be the mean field approximation and \( Y(t) = X(t) - x(t) \).

You can close the equations by assuming that \( Y^{(k)} = 0 \) for \( k > K \).

- For \( K = 0 \), this gives the mean field approximation (\( 1/N \)-accurate).
- For \( K = 2 \), this gives the refined mean field (\( 1/N^2 \)-accurate).
- For \( K = 4 \), this gives a second order expansion (\( 1/N^3 \)-accurate).

For a system of dimension \( d \), \( Y(t)^{(k)} \) has \( d^k \) equations.
Computational issues

- The mean field is a system of non-linear ODE of dimension $d$ (where $d = |S|$ if all agents have the same parameters and $N|S|$ if they are all different)
- The $1/N$ term adds two systems of time-inhomogeneous linear ODEs of dimension $d^2$ and $d$.
- The $1/N^2$ term adds four systems of time-inhomogeneous linear ODEs of dimension $d^4$, $d^3$, $d^2$ and $d$.

To compute, you essentially need up to the second (for the $1/N$-term) or the fourth (for the $1/N^2$-term) derivatives of the drifts.
We implemented this in a numerical library

https://github.com/ngast/rmf_tool/

The transitions are (for $i \in \{1 \ldots K\}$): 
\[
\begin{align*}
\frac{1}{N} e_i & \text{ rate } N \rho \left( x_{i-1}^2 - x_i^2 \right) \\
- \frac{1}{N} e_i & \text{ rate } N \left( x_i - x_{i+1} \right)
\end{align*}
\]
The refined mean field approximation is very accurate when predicting steady-state performance.

Arrival at each server $\rho$.

- Sample $d - 1$ other queues.
- Allocate to the shortest queue.

Service rate $= 1$.

<table>
<thead>
<tr>
<th></th>
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<th>$N = 20$</th>
<th>$N = 50$</th>
<th>$N = 100$</th>
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<tr>
<td>Mean Field</td>
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<td>2.3527</td>
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</tr>
<tr>
<td>$1/N$-expansion</td>
<td>2.7513</td>
<td>2.5520</td>
<td>2.4324</td>
<td>2.3925</td>
</tr>
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<td>2.8045</td>
<td>2.5653</td>
<td>2.4345</td>
<td>2.3930</td>
</tr>
<tr>
<td>Simulation</td>
<td>2.8003</td>
<td>2.5662</td>
<td>2.4350</td>
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Steady-state average queue length ($\rho = 0.9$).
The refined mean field approximation is very accurate
... to evaluate the transient performance

![Diagram showing random choice and selection]

Remark about computation time:

- 10min/1h (simulation $N = 1000$)
- Requires many simulations, confidence intervals...

Python numpy
The refined mean field approximation is very accurate... to evaluate the transient performance

Remark about computation time:

- Mean Field Approximation: 80ms
- 1/N-expansion: 700ms
- 1/N^2-expansion: 9s

10min/1h (simulation N = 1000/N = 10), C++ code. Requires many simulations, confidence intervals, ...
The refined mean field approximation is very accurate...
... to evaluate the transient performance

Remark about computation time:
- 10min/1h (simulation $N = 1000/N = 10$), C++ code. Requires many simulations, confidence intervals, ...
- 80ms (mean field), 700ms ($1/N$-expansion), 9s ($1/N^2$-expansion), Python numpy
The refined approximation can also account for behaviors that are indistinguishable by classical mean field methods.

Example: choosing with or without replacement

Let $x_i$ be the fraction of servers with $i$ or more jobs. Pick two servers, what is the probability that the least loaded has exactly $i$ jobs?

If picked with replacement:

$$x_i^2 - x_i^{i+1} + 1.$$ 

If picked without replacement:

$$x_i^{N+i-1} - rac{N+1}{N}x_i^{i+1} - rac{1}{N}.$$ 

The two coincide as $N \to \infty$.

Simulation

Refined mean field

$\rho = 0$.

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<tr>
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<th>Without</th>
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<tr>
<td></td>
<td>2.820</td>
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Example: choosing with or without replacement

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- If picked with replacement: $x_i^2 - x_{i+1}^2$.
- If picked without replacement: $x_i \frac{Nx_i - 1}{N - 1} - x_{i+1} \frac{Nx_{i+1} - 1}{N - 1}$.

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2. Moment closure and *refined* mean field approximation

3. Conclusion: Does it always work?
Recap and extensions

If you fix a control policy such that $x \mapsto xQ(x)$ is $C^2$, then:

1. The accuracy of the classical mean field approximation is $O(1/N)$.
   - Mean field approximation = propagation of chaos (= independence)
2. We can use this to define a refined approximation.
   - Refined mean field approximation = look at covariance
3. The refined approximation is often accurate for $N = 10$:
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Extensions:

- Transient regime
- Discrete-time systems
- We can also compute the next term in $1/N^2$. 
Limit 1: it applies to agent properties but not to populations

Population’s state: \( X(t) = \frac{1}{N} \sum_{n=1}^{N} \delta_{S_n(t)} \)

One agent has state \( S_n(t) \)

\[
X(t) = x(t) + \frac{G(t)}{\sqrt{N}}
\]

\[
\mathbb{E}[X(t)] = x(t) + \frac{C}{N}
\]

Average queue length

\((N = 10 \text{ and } \rho = 0.9)\)

<table>
<thead>
<tr>
<th>Simu</th>
<th>Refined M.F.</th>
<th>M.F.</th>
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<tbody>
<tr>
<td>2.804</td>
<td>2.751</td>
<td>2.353</td>
</tr>
</tbody>
</table>
Limit 2: It can fail when the mean field approximation has limiting cycles

Transition

\[(D, A, S) \mapsto (D - \frac{1}{N}, A + \frac{1}{N}, S)\]
\[(D, A, S) \mapsto (D, A - \frac{1}{N}, S + \frac{1}{N})\]
\[(D, A, S) \mapsto (D + \frac{1}{N}, A, S - \frac{1}{N})\]

Rate

\[N(0.1 + 10X_A)X_D\]
\[N5X_A\]
\[N(1 + \frac{10X_A}{X_D + \delta})X_S\]
Limit 2: It can fail when the mean field approximation has limiting cycles
Limit 2: It can fail when the mean field approximation has limiting cycles
Limit 3: What about games and/or optimal control?

Discrete-state mean field games are relatively “easy” to work with.

- Forward equation : ODE.
- Backward equation : MDP (Markov decision process)

Open question : Do the Nash equilibria of the finite games converge to a mean field equilibria? What is the rate of convergence?

There are examples with refined-$1/N$ equilibrium (see Gueant et al. “when does the meeting start”). The value of the game does not always converge (Doncel et al. 2017) When it does, convergence is often at rate $O(1/\sqrt{N})$. 

Nicolas Gast – 28 / 29
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Some References

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