Bilevel Programming and Price Optimization

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Inria INVENTORS FOR THE DIGITAL WORLD

ULB

Bilevel Program

$$
\begin{array}{rl}
\max\limits_{x,y} & f(x,y) \\
\text{s.t.} & (x,y) \in X \\
& y \in S(x) \\
\text{where} & S(x) = \operatorname*{argmax}\limits_{y} g(x,y) \\
& \text{s.t.} (x,y) \in Y
\end{array}
$$

Adequate framework for Stackelberg game

- Leader: 1st level,
- Follower: 2nd level,
- Leader takes follower's optimal reaction into account.

Heinrich von Stackelberg
(1905 - 1946)

First paper on bilevel optimization

Bracken & McGill (OR, 1973): First bilevel model, structural properties, military application.

Mathematical Programs with Optimization Problems in the Constraints

Jerome Bracken and James T. McGill

Institute for Defense Analyses, Arlington, Virginia

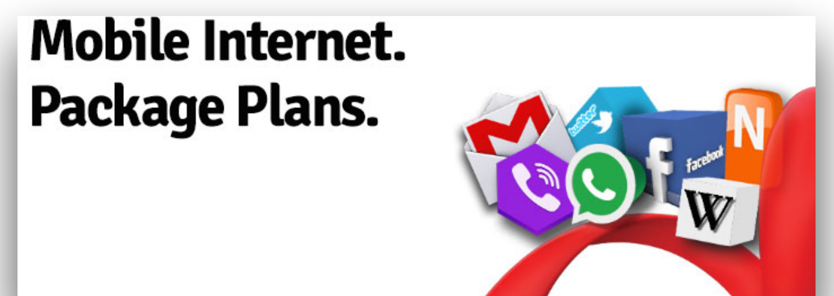
(Received October 5, 1971)

This paper considers a class of optimization problems characterized by constraints that themselves contain optimization problems. The problems in the constraints can be linear programs, nonlinear programs, or two-sided optimization problems, including certain types of games. The paper presents theory dealing primarily with properties of the relevant functions that result in convex programming problems, and discusses interpretations of this theory. It gives an application with linear programs in the constraints, and discusses computational methods for solving the problems.

Adequate framework for Price Setting Problem

$$\max_{T \in \Theta, x, y} \quad F(T, x, y)$$
$$\text{s.t.} \quad \min_{x,y} f(T, x, y)$$
$$\text{s.t.} (x, y) \in \Pi$$

Applications

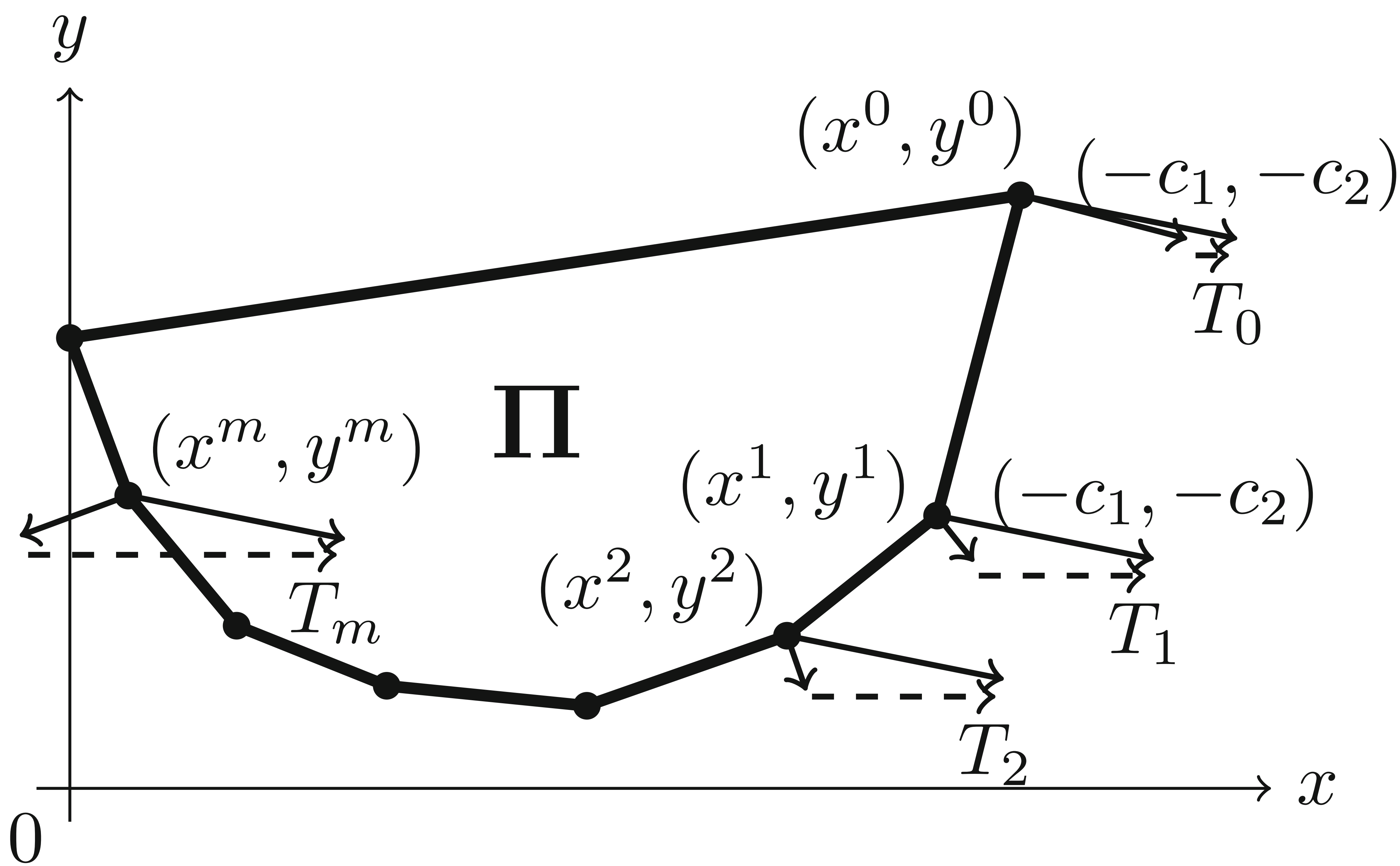


Price Setting Problem with linear constraints

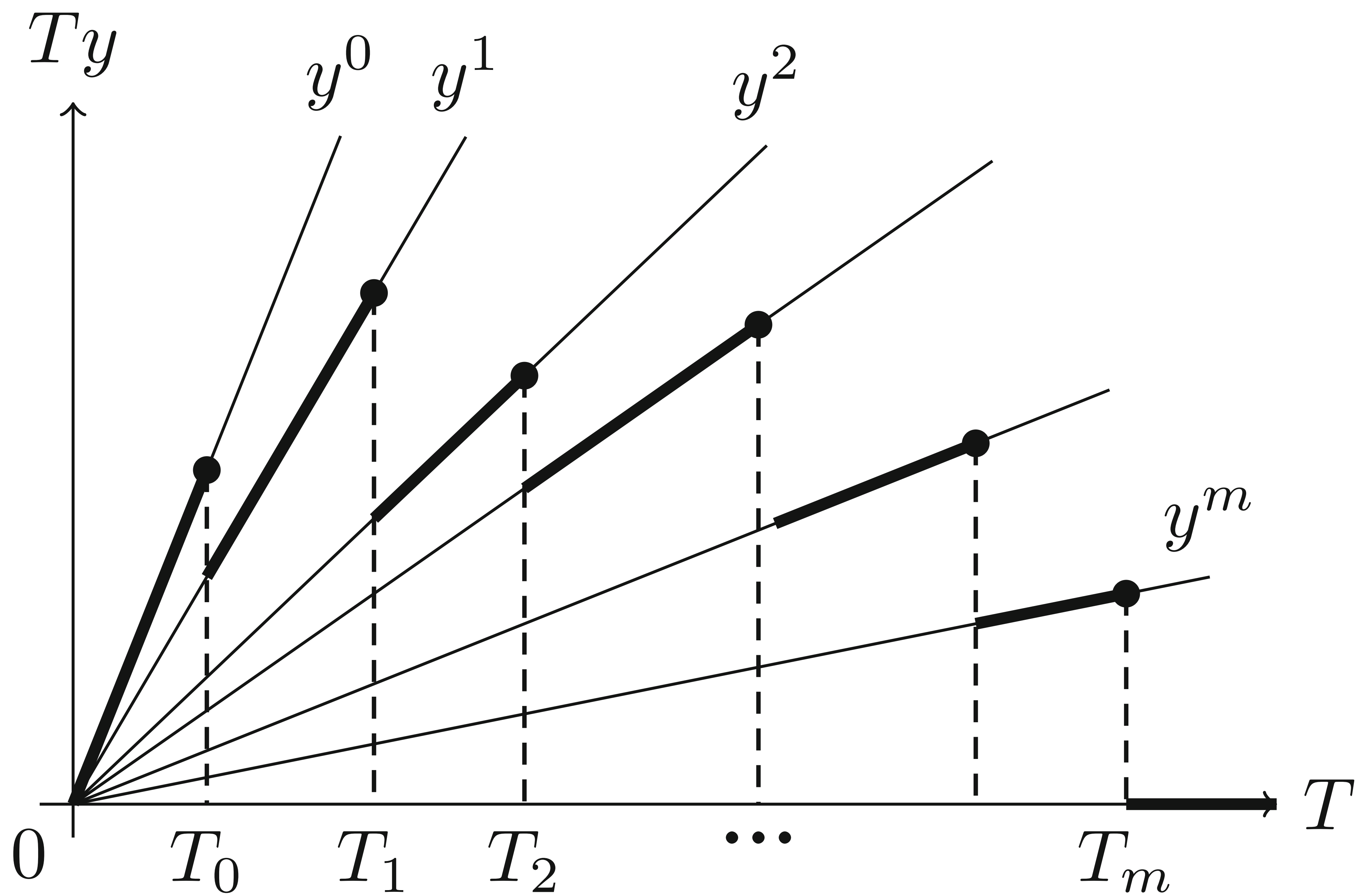
$$
\begin{aligned}
\max_{T,x,y} \quad & Tx \\
\text{s.t.} \quad & TC \geq f \\
\min_{x,y} \quad & (c+T)x + dy \\
\text{s.t.} \quad & Ax + By \geq b
\end{aligned}
$$

- $\Pi = \{x, y : Ax + By \geq b\}$ is bounded
- $\{(x, y) \in \Pi : x = 0\}$ is nonempty

Example: 2 variables in second level



The first level revenue



Price setting problem: single level reformulation

$$
\begin{aligned}
\max_{T,x,y} \quad & Tx \\
\text{s.t.} \quad & TC \geq f \\
\min_{x,y} \quad & (c+T)x + dy \\
\text{s.t.} \quad & Ax + By \geq b
\end{aligned}
$$

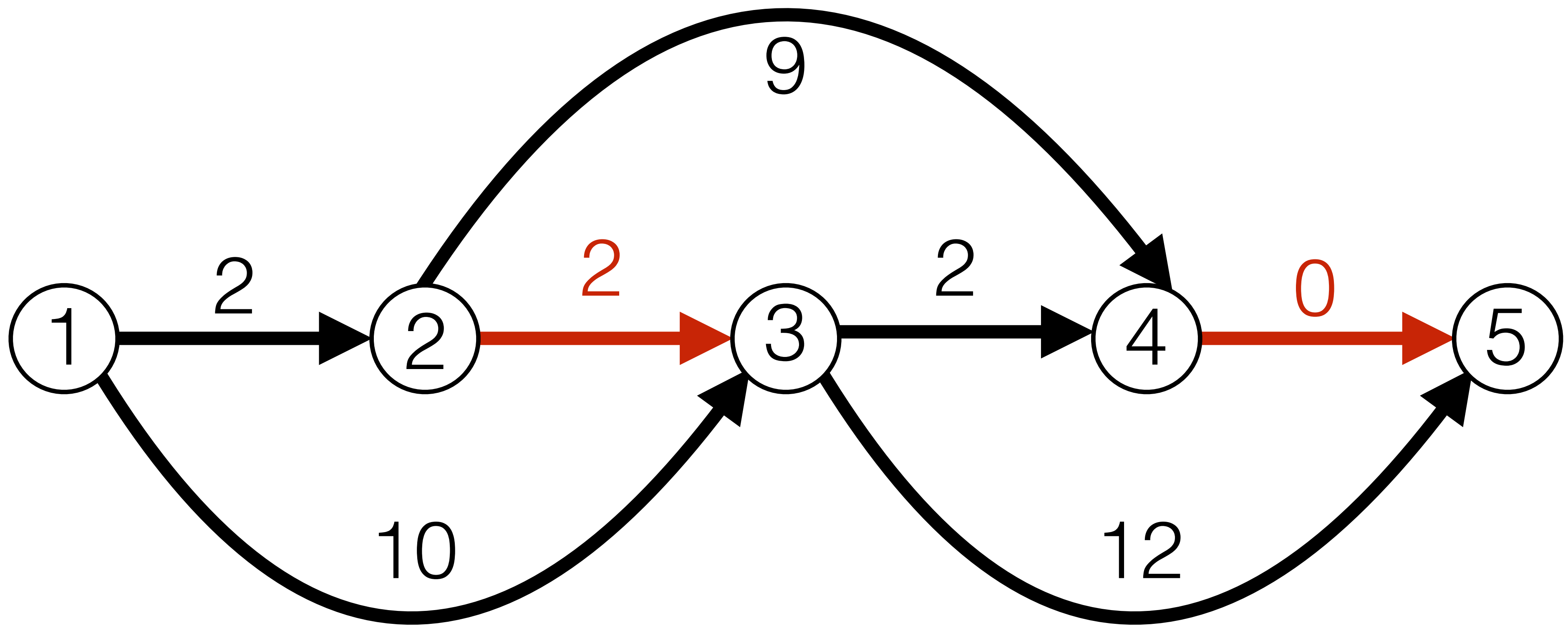
$$
\begin{aligned}
\max_{T,x,y} \quad & Tx \\
\text{s.t.} \quad & TC \geq f \\
& Ax + By \geq b \\
& \lambda A = c + T \\
& \lambda B = d \\
& (c+T)x + dy = \lambda b
\end{aligned}
$$

Network pricing problem

(Labbé et al. 1998, Labbé & Violin, 2013)

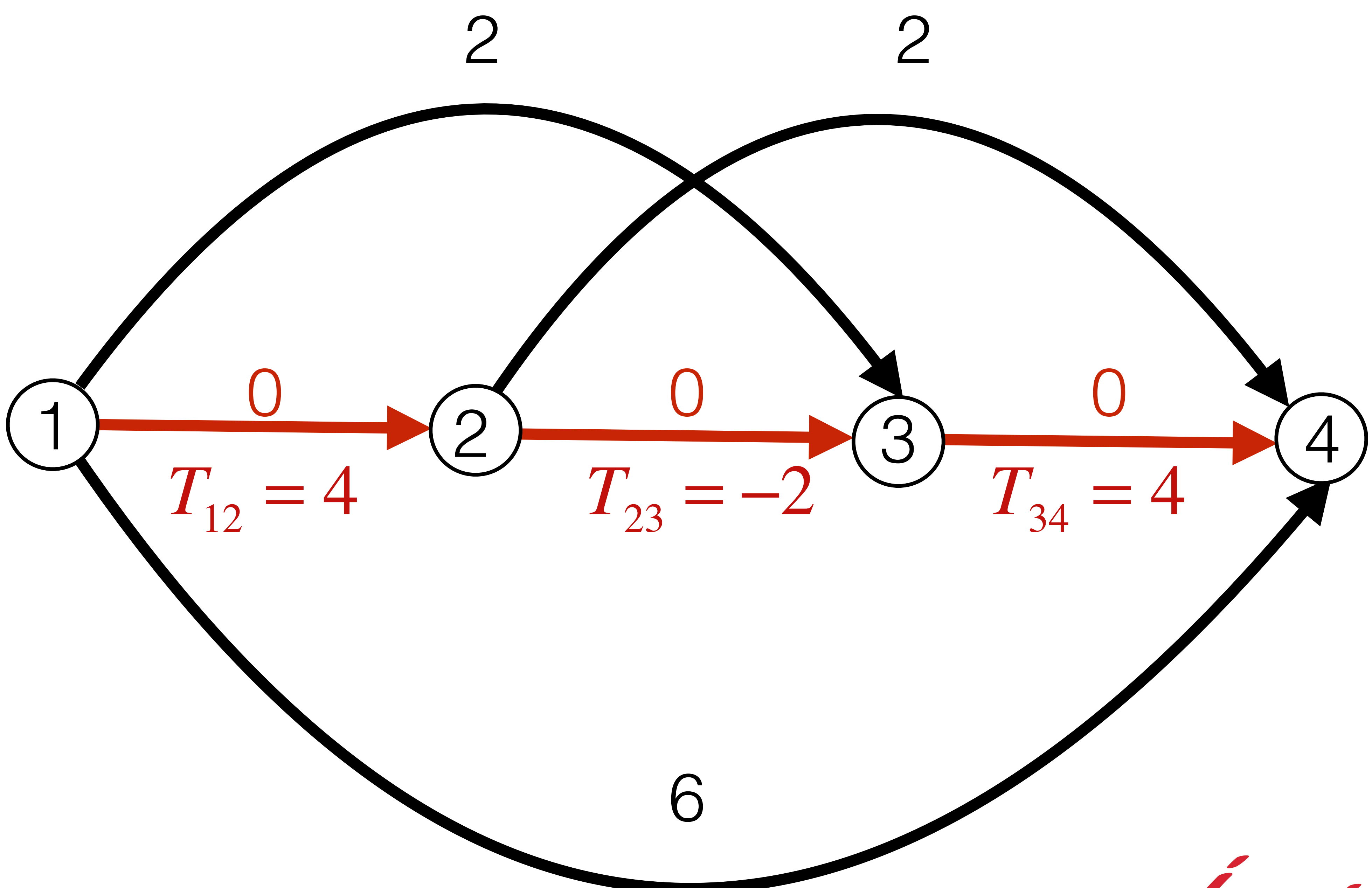
- network with toll arcs (A_1) and non toll arcs (A_2)
- Costs c_a on arcs
- Commodities (o^k, d^k, n^k)
- Routing on cheapest (cost + toll) path
- Maximize total revenue

Example



- UB on $(T_1 + T_2) = \mathrm{SPL}(T = \infty) - \mathrm{SPL}(T = 0) = 22 - 6 = 16$
- $T_{2,3} = 5, T_{4,5} = 10$

Example with negative toll arc



Network pricing problem

(Labbé et al., 1998, Roch at al., 2005)

- Strongly NP-hard even for only one commodity.
- Polynomial for
  - one commodity if lower level path is known
  - one commodity if toll arcs with positive flows are known
  - one single toll arc.
- Polynomial algorithm with worst-case guarantee of $(\log |A1|)/2 + 1$

One toll arc

- For each k, compute $UB(k)$ on profit(k) if k uses toll arc

- $UB(1) \geq UB(2) \geq \ldots \geq UB(K)$

- $T_a = UB(i^*), i^* \in \operatorname*{argmax}_i \{UB(i) \sum_{k \leq i} n^k\}$

Network pricing problem

$$\max_{T} \quad \sum_{a \in A_1} T_a \sum_{k \in K} n^k x_a^k$$

$$\min_{x,y} \quad \sum_{k \in K} \Big(\sum_{a \in A_1} (c_a + T_a) x_a^k + \sum_{a \in A_2} c_a y_a^k \Big)$$

$$\text{s.t.} \quad \sum_{a \in i^+} (x_a^k + y_a^k) - \sum_{a \in i^-} (x_a^k + y_a^k) = b_i^k \quad \forall k, i$$

$$x_a^k, y_a^k \geq 0, \quad \forall k, a$$

NPP: single level reformulation

$$
\begin{aligned}
\max_{T,x,y,\lambda} \quad & \sum_{k \in K} n^k \sum_{a \in A_k} T_a x_a^k \\
\text{s.t.} \quad & \sum_{a \in i^+} (x_a^k + y_a^k) - \sum_{a \in i^-} (x_a^k + y_a^k) = b_i^k \quad \forall k, i \\
& \lambda_i^k - \lambda_j^k \leq c_a + T_a \quad \forall k, a \in A_1, i, j \\
& \lambda_i^k - \lambda_j^k \leq c_a \quad \forall k, a \in A_2, i, j \\
& \sum_{a \in A_1} (c_a + T_a) x_a^k + \sum_{a \in A_2} c_a y_a^k = \lambda_{o^k}^k - \lambda_{d^k}^k \\
& x_a^k, y_a^k \geq 0 \quad \forall k, a \\
& T_a \geq 0 \quad \forall a \in A_1
\end{aligned}
$$

NPP: single level reformulation

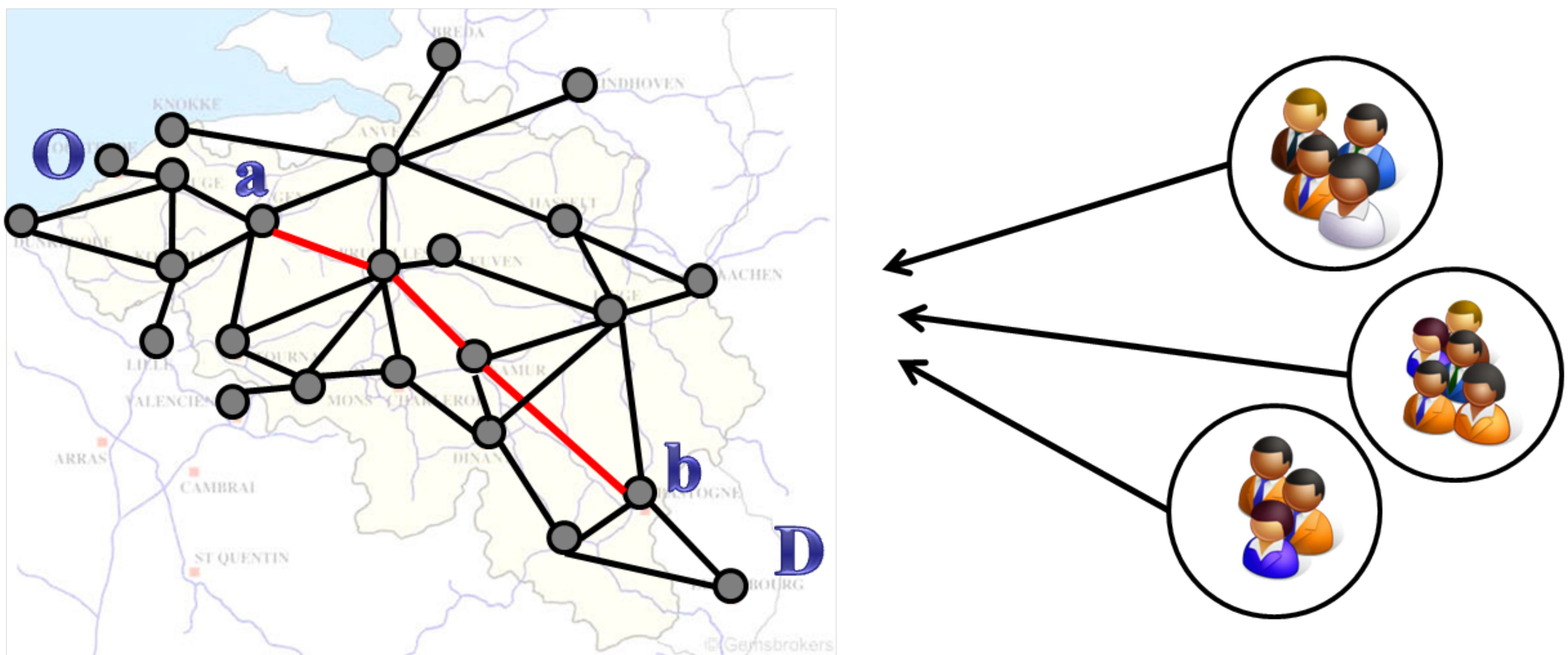
$$
\begin{aligned}
\max_{T,x,y,\lambda} \quad & \sum_{k \in K} n^k \sum_{a \in A_k} T_a x_a^k \\
\text{s.t.} \quad & \sum_{a \in i^+} (x_a^k + y_a^k) - \sum_{a \in i^-} (x_a^k + y_a^k) = b_i^k \quad \forall k, i \\
& \lambda_i^k - \lambda_j^k \leq c_a + T_a \quad \forall k, a \in A_1, i, j \\
& \lambda_i^k - \lambda_j^k \leq c_a \quad \forall k, a \in A_2, i, j \\
& \sum_{a \in A_1} (c_a + T_a) x_a^k + \sum_{a \in A_2} c_a y_a^k = \lambda_{o^k}^k - \lambda_{d^k}^k \\
& x_a^k, y_a^k \geq 0 \quad \forall k, a \\
& T_a \geq 0 \quad \forall a \in A_1
\end{aligned}
$$

NPP: obtaining a MIP

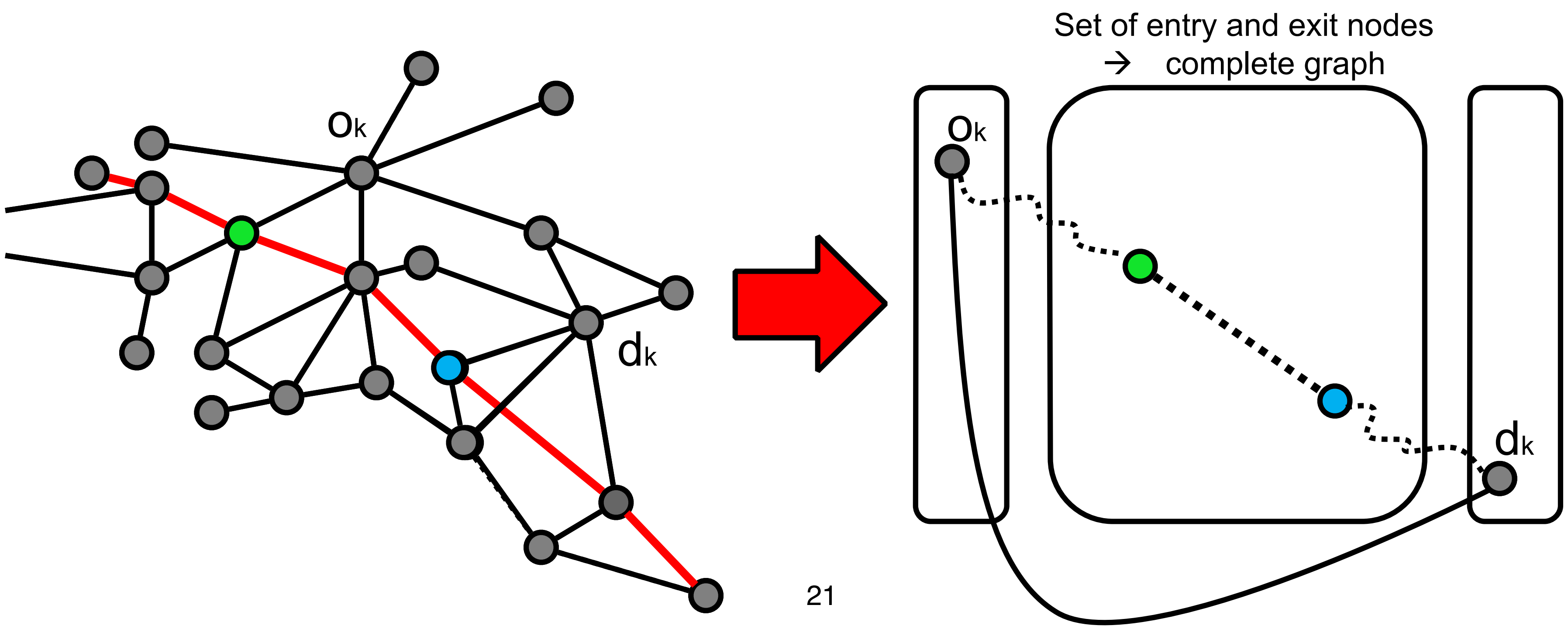
$$T_a x_a^k = p_a^k$$

$$\boxed{\begin{aligned} p_a^k &\leq M_a^k x_a^k \\ T_a - p_a^k &\leq N_a(1 - x_a^k) \\ p_a^k &\leq T_a \\ x_a^k &\in \{0,1\} \end{aligned}}$$

Particular case: highway pricing

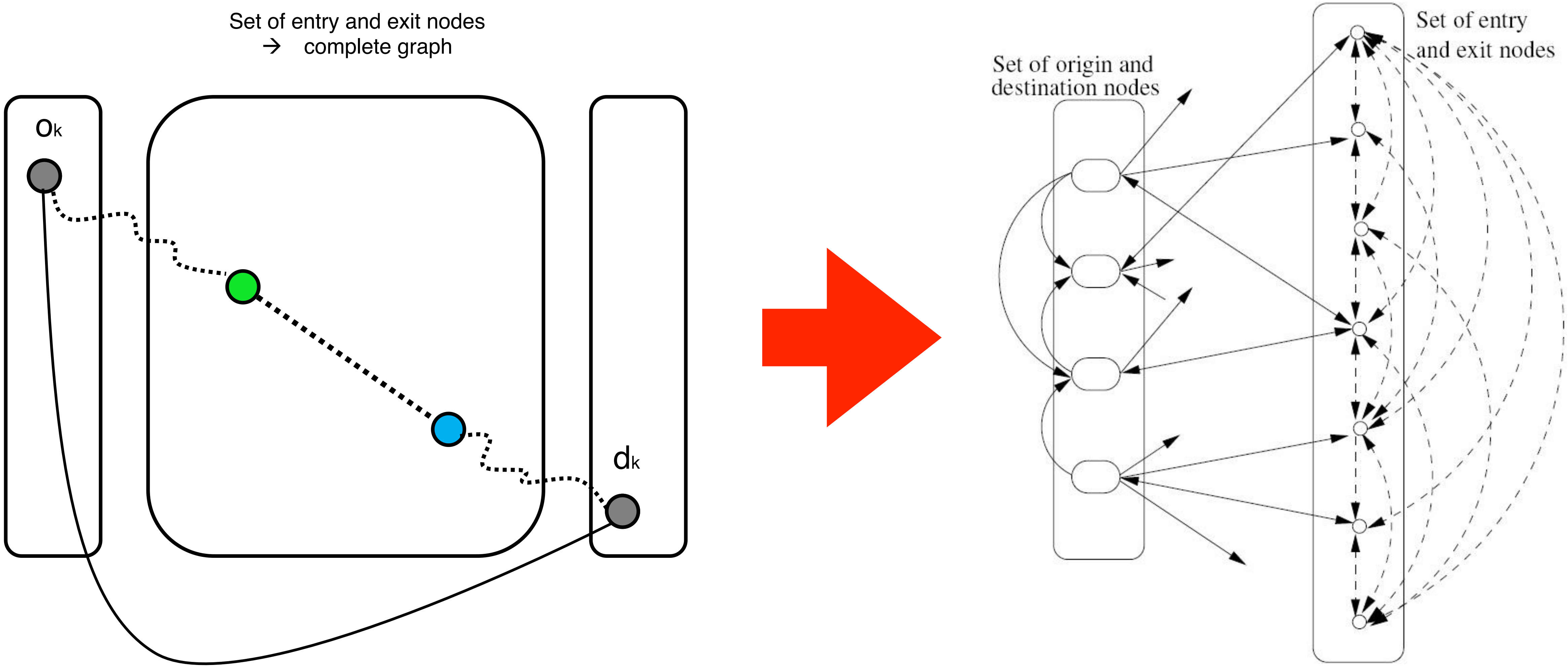


Particular case: highway pricing

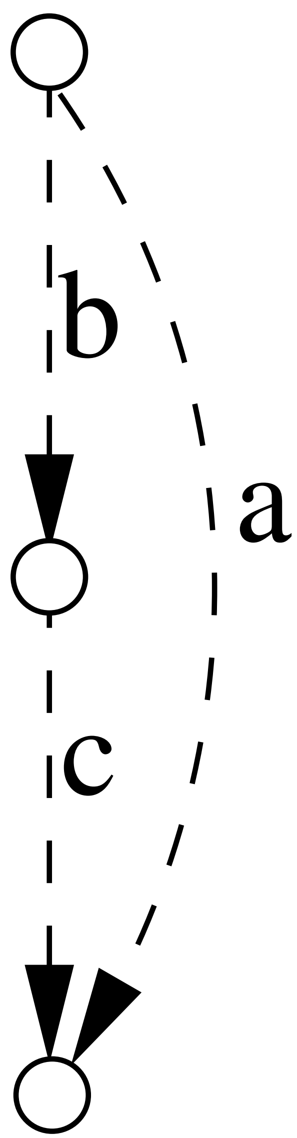


- Polynomial number of paths for commodities
- Tolls non additive: one toll for each path

Particular case: highway pricing

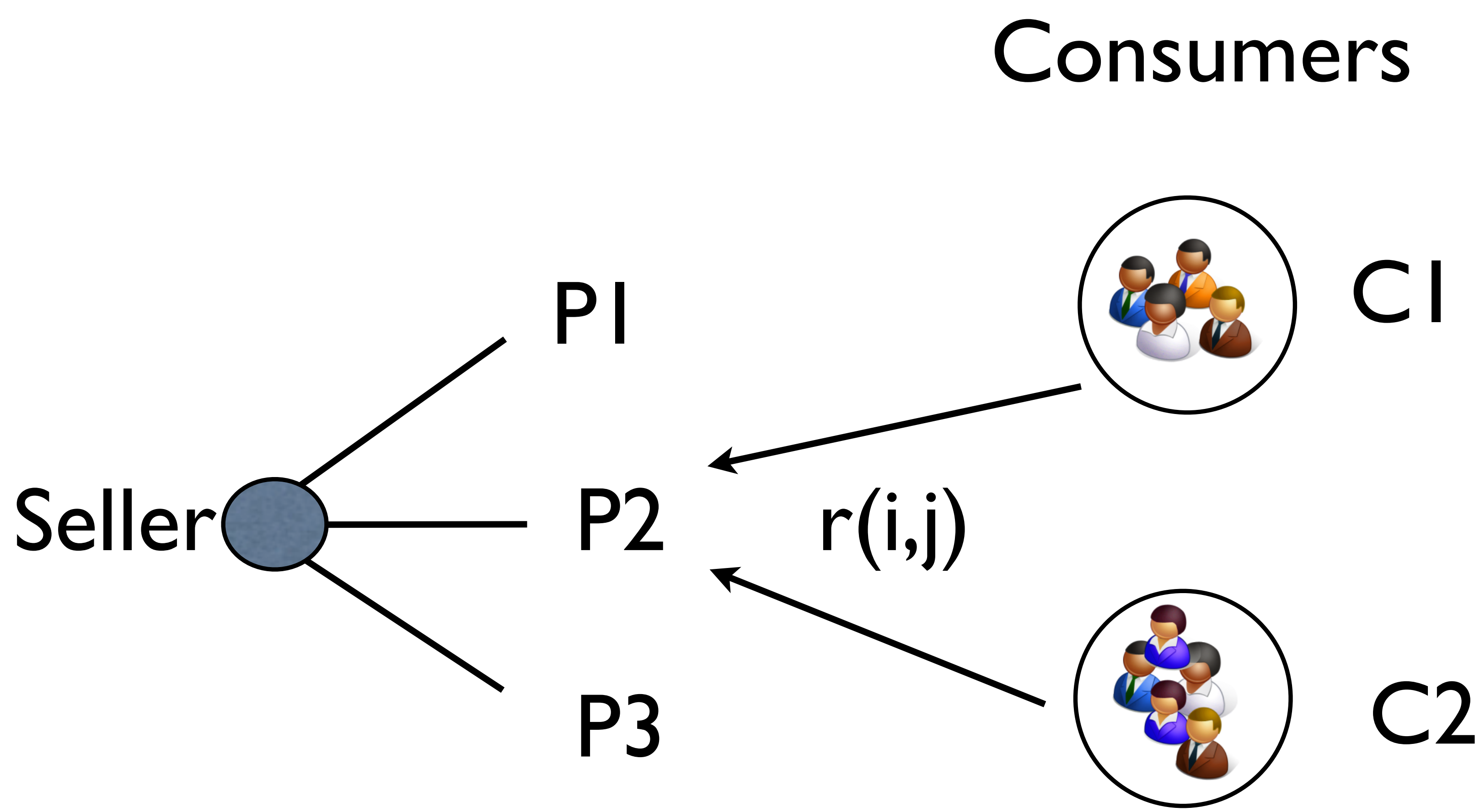


Possible additional constraints



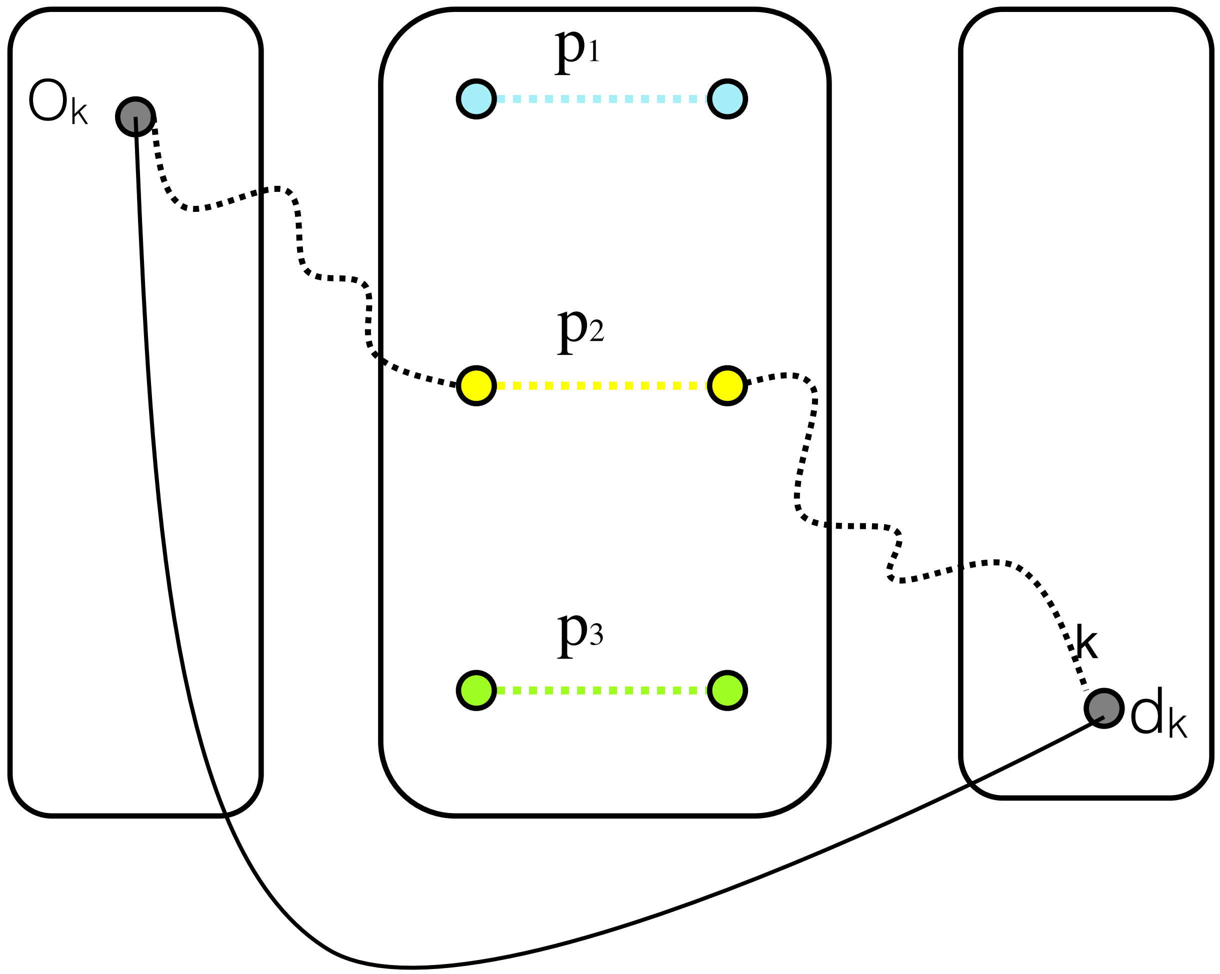
$$
\begin{aligned}
T_a &\leq T_b + T_c \\
T_a &\geq T_b
\end{aligned}
$$

An equivalent problem: Product pricing



- p_i = price of product i
- $r(i,j)$ = reservation price of consumer group C_j for product i

Product pricing (PPP)



$$r(i,k) = c^k_{o,d} - c^k_i$$

PPP= bilevel formulation

$$
\begin{aligned}
\max_{T \geq 0} \quad & \sum_{k \in K} n^k \sum_{a \in A^k} T_a x_a^k \\
\text{s.t.} (x, y) \in \operatorname*{argmin}_{x,y} \quad & \sum_{k \in K} \Big(\sum_{a \in A^k} (c_a + T_a) x_a^k + c_{od}^k y^k \Big) \\
\text{s.t.} \quad & \sum_{a \in A^k} x_a^k + y^k = 1, \forall k \in K \\
& x_a^k, y^k \in \{0, 1\}
\end{aligned}
$$

PPP: single level formulation

$$\max_{T \geq 0} \quad \sum_{k \in K} n^k \sum_{a \in A^k} T_a x_a^k$$

$$\text{s.t.} \quad \sum_{a \in A^k} (c_a^k + T_a) x_a^k + c_{od}^k y^k \leq T_b + c_b^k, \quad \forall k, b$$

$$\sum_{a \in A^k} (c_a^k + T_a) x_a^k + c_{od}^k y^k \leq c_{od}^k, \quad \forall k$$

$$\sum_{a \in A^k} x_a^k + y^k = 1 \quad \forall k$$

$$x_a^k, y^k \in \{0, 1\}$$

PPP: MIP formulation

(Heilporn et al., 2010, 2011)

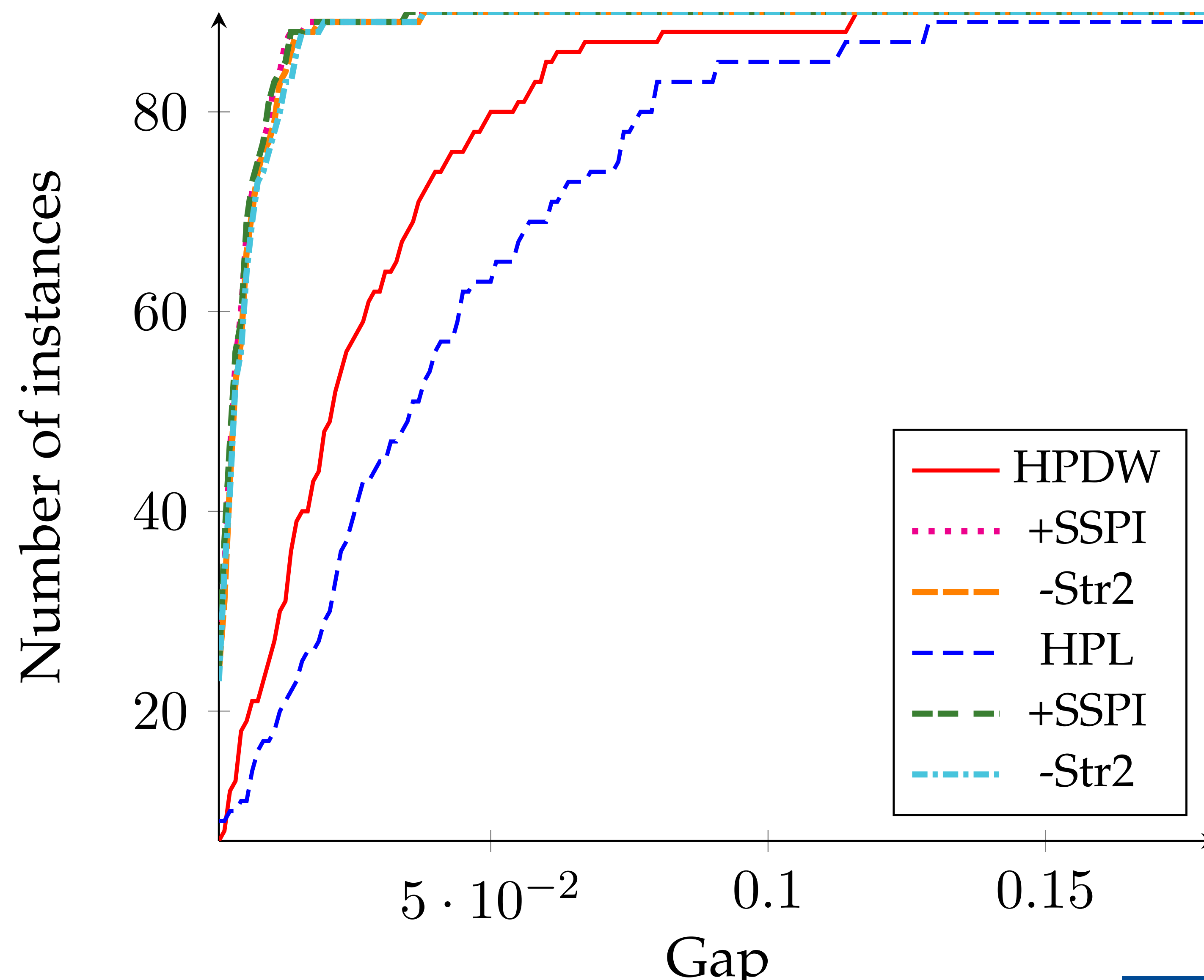
$$
\begin{aligned}
\max \quad & \sum_{k \in K} n^k \sum_{a \in A^k} p_a^k \\
\text{s.t.} \quad & \sum_{a \in A^k} (c_a^k x_a^k + p_a^k) + c_{od}^k y^k \leq T_b + c_b^k, \quad \forall k, b \\
& \sum_{a \in A^k} (c_a^k x_a^k + p_a^k) + c_{od}^k y^k \leq c_{od}^k, \quad \forall k \\
& \sum_{a \in A^k} x_a^k + y^k = 1 \quad \forall k \\
& p_a^k \leq M_a^k x_a^k \quad \forall k, a \\
& T_a - p_a^k \leq N_a (1 - x_a^k) \quad \forall k, a \\
& 0 \leq p_a^k \leq T_a \quad \forall k, a \\
& x_a^k, y^k \in \{0, 1\}
\end{aligned}
$$

PPP: MIP formulation

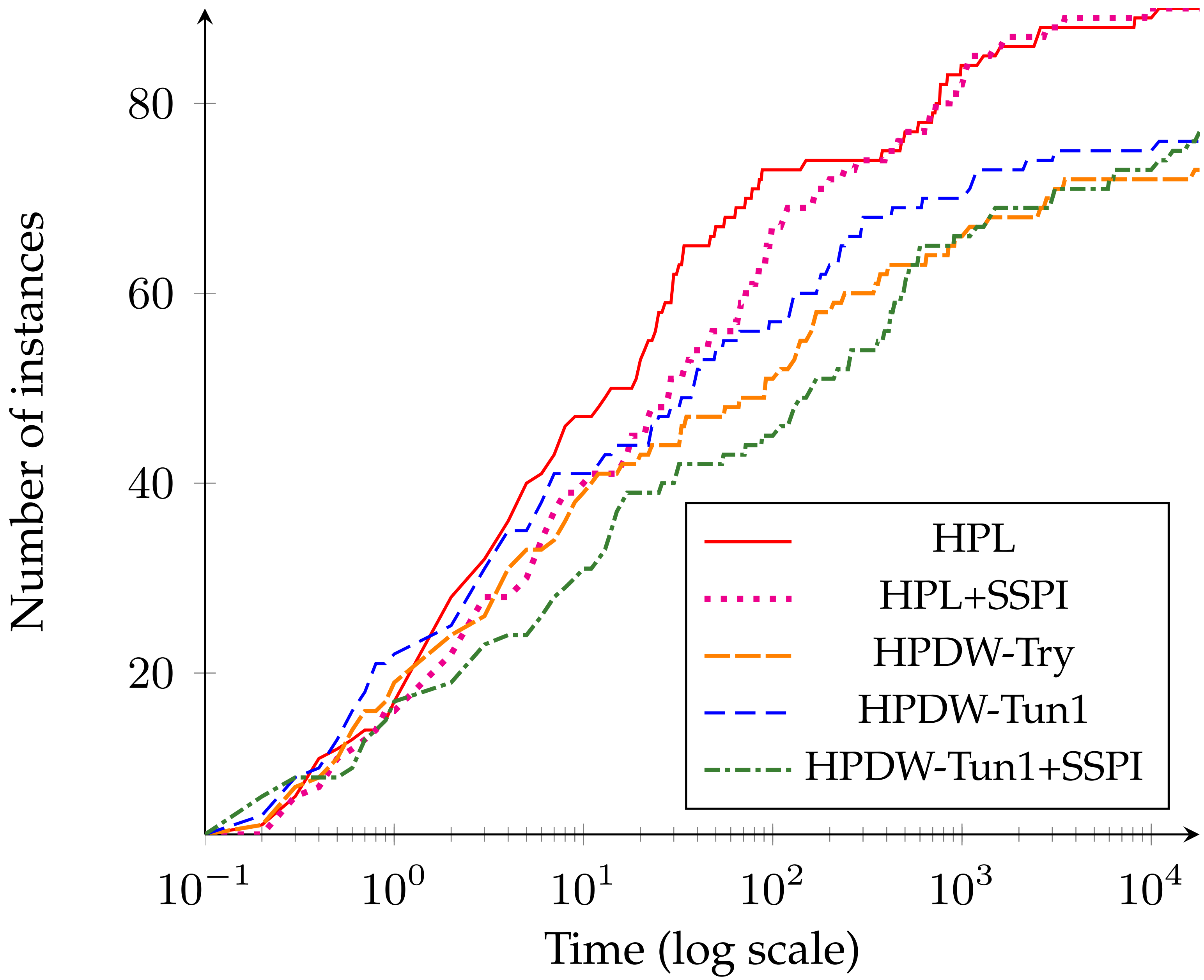
- Strengthening $\sum_{a \in A^k} (c_a^k x_a^k + p_a^k) + c_{od}^k y^k \leq T_b + c_b^k \Rightarrow$ facet and divides gap by 2
- LP-relaxation(strengthened formulation) = ideal formulation for one commodity

PPP: gap
(Violin, 2014)

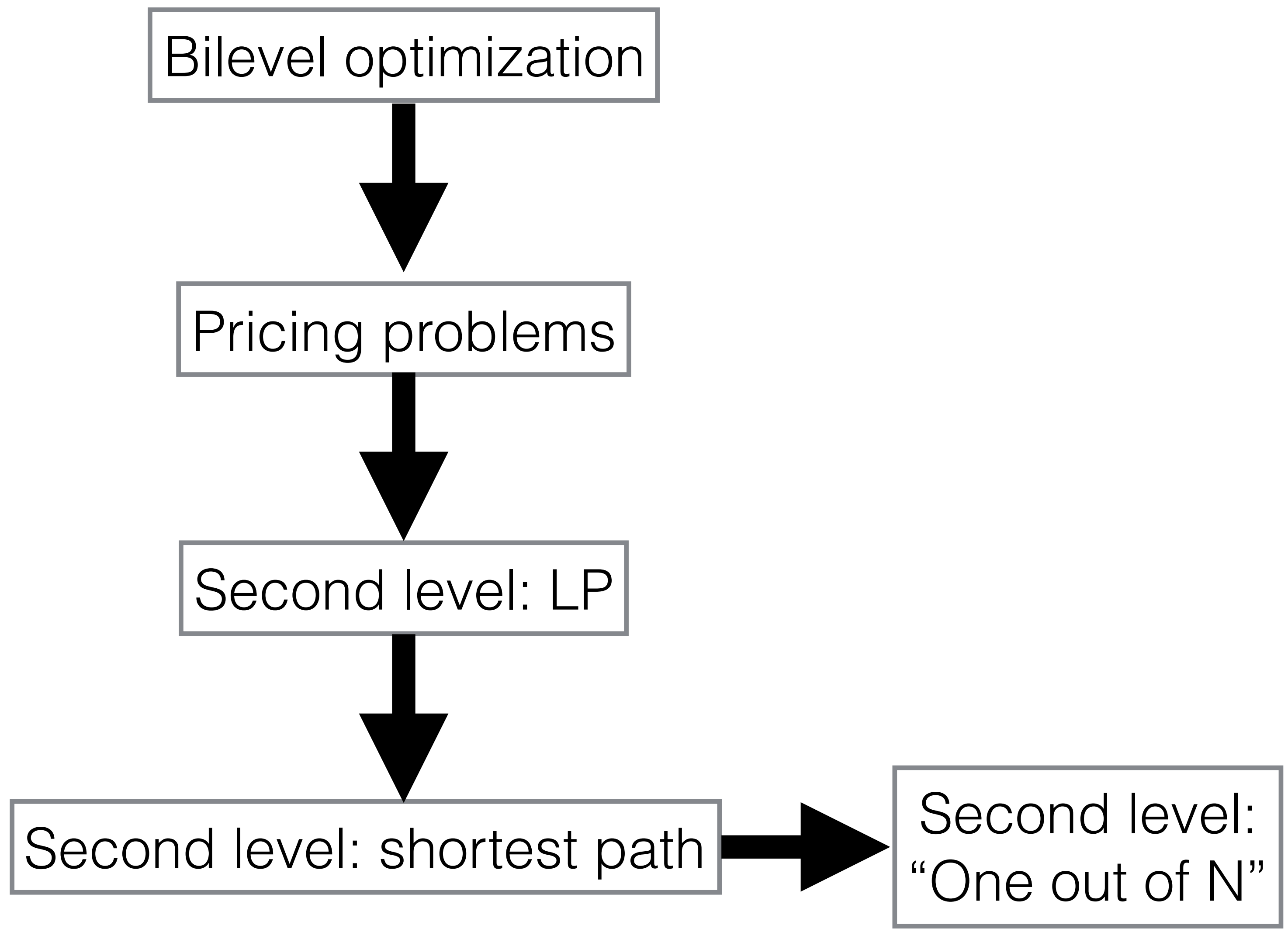
20 - 90 arcs
20 - 90 commodities



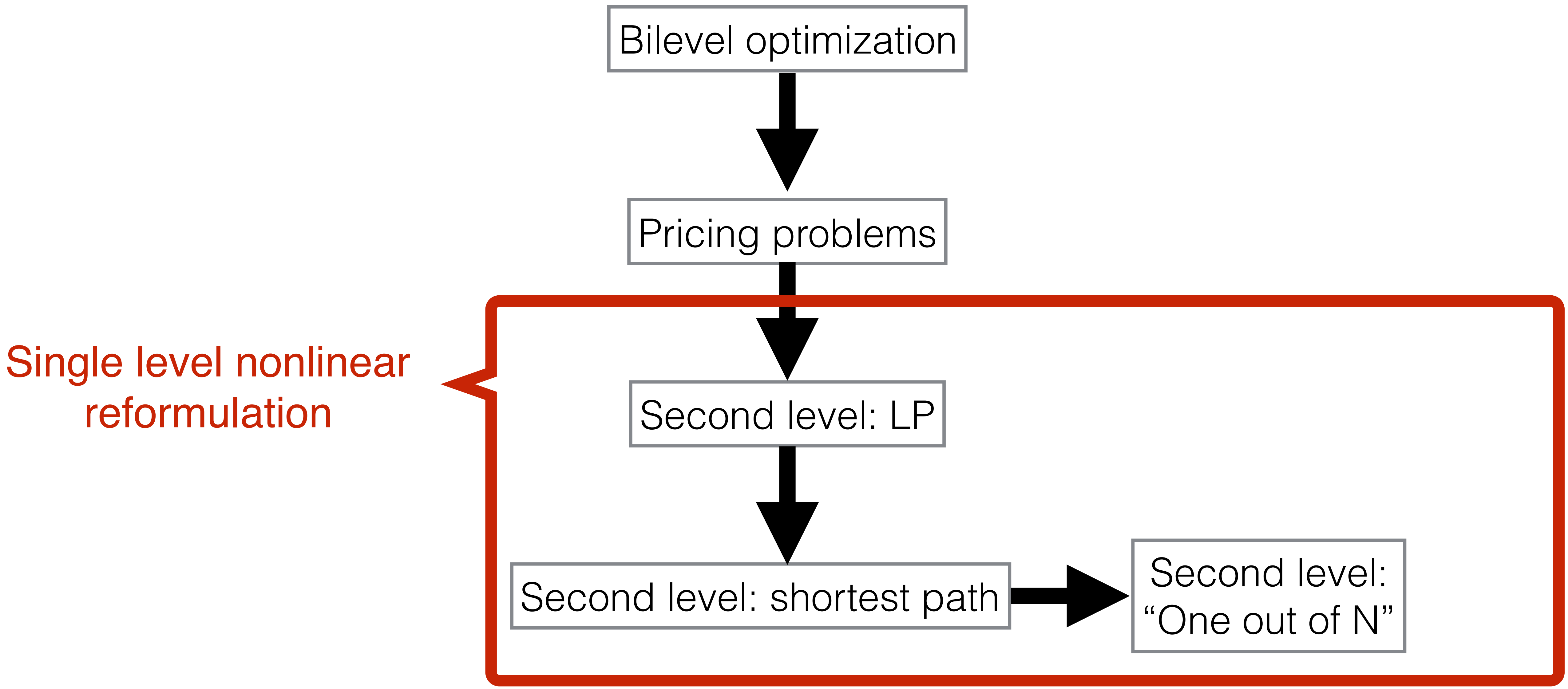
PPP: computing time



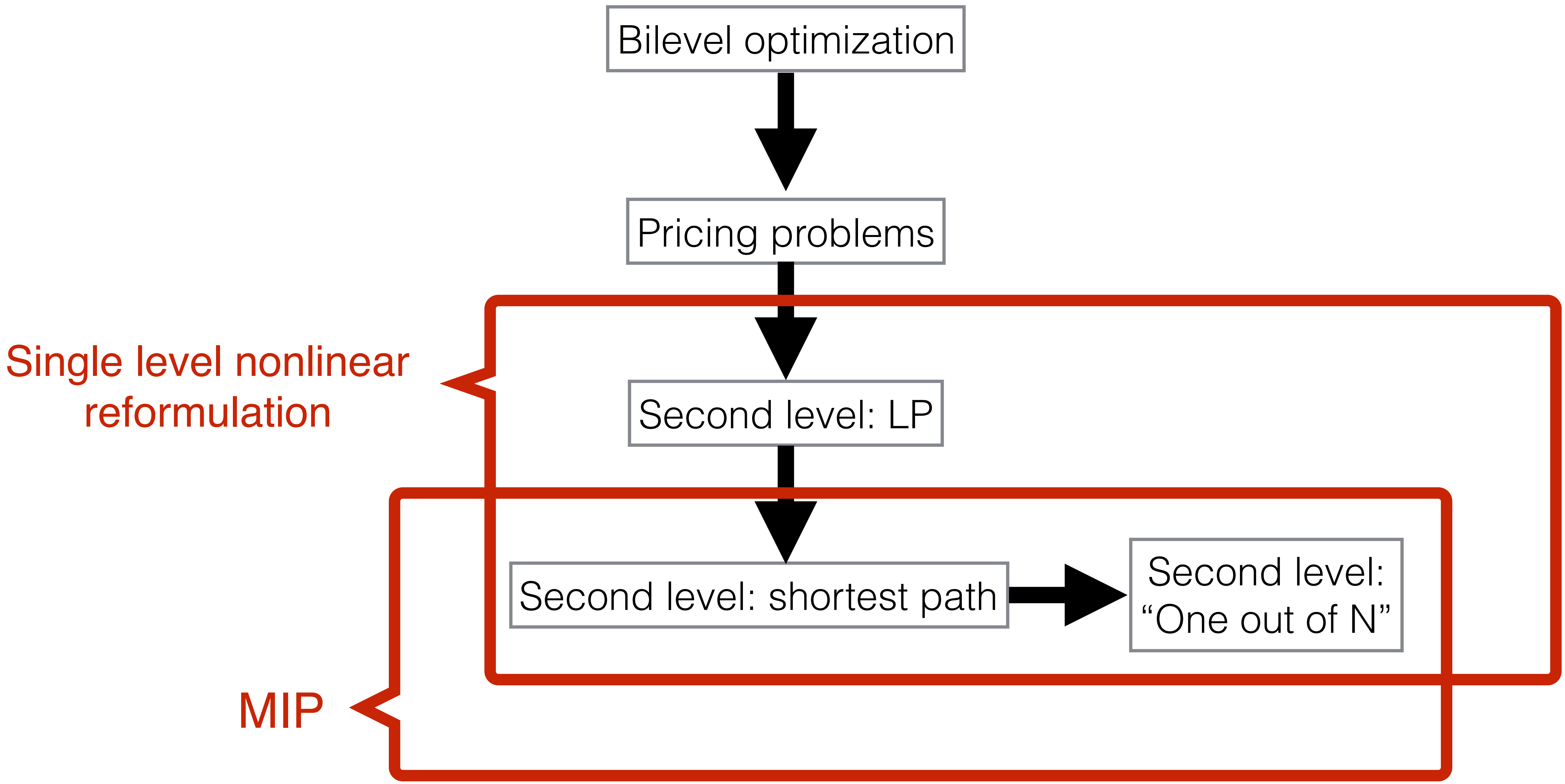
RECAP


Bilevel optimization
Pricing problems
Second level: LP
Second level: shortest path
Second level: “One out of N”

RECAP


Bilevel optimization
Pricing problems
Single level nonlinear reformulation
Second level: LP
Second level: shortest path
Second level: “One out of N”

RECAP


Bilevel optimization
Pricing problems
Single level nonlinear reformulation
Second level: LP
Second level: shortest path
Second level: "One out of N"
MIP

Conclusion

- Bilevel model: rich framework for pricing in network-based industries.
- Models: theoretically and computationally challenging.
- Need to exploit problem's inner structure.
- Analysis of basic model: relevant and useful for attacking real applications.
- Integration of real-life features (congestion, market segmentation, dynamics, uncertainty...).
- Investigate variants of product pricing: rank pricing, single minded customers, bundle pricing, etc. See my Google page.


THANK YOU


Questions?